

GS-2015

(Computer & Systems Sciences)

**TATA INSTITUTE OF FUNDAMENTAL RESEARCH**

Written Test in **COMPUTER & SYSTEMS SCIENCES - December 14, 2014**

Duration : Three hours (3 hours)

Name : \_\_\_\_\_ Ref. Code : \_\_\_\_\_

**Please read all instructions carefully before you attempt the questions.**

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine readable. Use only Black/Blue ball point pen to fill-in the answer sheet.
2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.
3. This question paper consists of three (3) parts. **Part-A** contains fifteen (15) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain fifteen (15) questions each, directed towards candidates for (B) Computer Science and (C) Systems Science (including Communications & Math Finance), respectively. **STUDENTS MAY ATTEMPT EITHER PART-B OR PART-C.** **In case, a student attempts both Parts B & C (no extra time will be given) and qualifies for interview in both B & C, he/she will have opportunity to be interviewed in both areas.** All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks.
4. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.
5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.
6. **Use of calculators is NOT permitted.**
7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

## Part A Common Part

1. Consider a 6-sided die with all sides not necessarily equally likely such that probability of an even number is  $P(\{2, 4, 6\}) = 1/2$ , probability of a multiple of 3 is  $P(\{3, 6\}) = 1/3$ , and probability of 1 is  $P(\{1\}) = 1/6$ . Given the above conditions, choose the strongest (most stringent) condition of the following that must always hold about  $P(\{5\})$ , the probability of 5.

- (a)  $P(\{5\}) = 1/6$
- (b)  $P(\{5\}) \geq 1/6$
- (c)  $P(\{5\}) \leq 1/6$
- (d)  $P(\{5\}) \leq 1/3$  ✓
- (e) none of the above

2. Consider a circle with a circumference of one unit length. Let  $d < 1/6$ . Suppose that we independently throw two arcs, each of length  $d$ , randomly on this circumference so that each arc is uniformly distributed along the circle circumference. The arc attaches itself exactly to the circumference so that arc of length  $d$  exactly covers length  $d$  of the circumference. What can be said about the probability that the two arcs do not intersect each other?

- (a) It equals  $(1 - d)$
- (b) It equals  $(1 - 3d)$
- (c) It equals  $(1 - 2d)$  ✓
- (d) It equals 1
- (e) It equals  $(1 - d)(1 - d)$

3. Let  $|z| < 1$ . Define  $M_n(z) = \sum_{i=1}^{10} z^{10^n(i-1)}$ ? What is

$$\prod_{i=0}^{\infty} M_i(z) = M_0(z) \times M_1(z) \times M_2(z) \times \dots?$$

- (a) Can't be determined
- (b)  $1/(1 - z)$  ✓
- (c)  $1/(1 + z)$
- (d)  $1 - z^9$
- (e) None of the above

4. The Boolean function obtained by adding an inverter to each and every input of an AND gate is:

- (a) OR
- (b) XOR
- (c) NAND
- (d) NOR ✓
- (e) None of the above

5. What is logically equivalent to “If Kareena and Parineeti go to the shopping mall then it is raining”:
- (a) If Kareena and Parineeti do not go to the shopping mall then it is not raining.
  - (b) If Kareena and Parineeti do not go to the shopping mall then it is raining.
  - (c) If it is raining then Kareena and Parineeti go to the shopping mall.
  - (d) If it is not raining then Kareena and Parineeti do not go to the shopping mall. ✓
  - (e) None of the above
6. Ram has a fair coin, i.e., a toss of the coin results in either head or tail and each event happens with probability exactly half ( $1/2$ ). He repeatedly tosses the coin until he gets heads in two consecutive tosses. The expected number of coin tosses that Ram does is
- (a) 2
  - (b) 4
  - (c) 6 ✓
  - (d) 8
  - (e) None of the above
7. A  $1 \times 1$  chessboard has one (1) square, a  $2 \times 2$  chessboards has five (5) squares. Continuing along this fashion, what is the number of squares on the (regular)  $8 \times 8$  chessboard?
- (a) 64
  - (b) 65
  - (c) 204 ✓
  - (d) 144
  - (e) 256
8. There is a set of  $2n$  people:  $n$  male and  $n$  female. A good party is one with equal number of males and females (including the one where none are invited). The total number of good parties is
- (a)  $2^n$
  - (b)  $n^2$
  - (c)  $\binom{n}{\lfloor n/2 \rfloor}^2$
  - (d)  $\binom{2n}{n}$  ✓
  - (e) none of the above

9. Consider a square of side length 2. We throw five points into the square. Consider the following statements:

- (i) There will always be three points that lie on a straight line.
- (ii) There will always be a line connecting a pair of points such that two points lie on one side of the line and one point on the other.
- (iii) There will always be a pair of points which are at distance at most  $\sqrt{2}$  from each other.

Which of the above is true:

- (a) (i) only
- (b) (ii) only
- (c) (iii) only ✓
- (d) (ii) and (iii)
- (e) None of the above

10. Let  $f(x)$ ,  $x \in [0, 1]$ , be any positive real valued continuous function. Then

$$\lim_{n \rightarrow \infty} (n+1) \int_0^1 x^n f(x) dx$$

equals

- (a)  $\max_{x \in [0,1]} f(x)$
- (b)  $\min_{x \in [0,1]} f(x)$
- (c)  $f(0)$
- (d)  $f(1)$  ✓
- (e)  $\infty$

11. Suppose that  $f(x)$  is a continuous function such that  $0.4 \leq f(x) \leq 0.6$  for  $0 \leq x \leq 1$ . Which of the following is always true?

- (a)  $f(0.5) = 0.5$ .
- (b) There exists  $x$  between 0 and 1 such that  $f(x) = 0.8x$ . ✓
- (c) There exists  $x$  between 0 and 0.5 such that  $f(x) = x$ .
- (d)  $f(0.5) > 0.5$ .
- (e) None of the above statements are always true.

12. Consider two independent and identically distributed random variables  $X$  and  $Y$  uniformly distributed in  $[0, 1]$ . For  $\alpha \in [0, 1]$ , the probability that  $\alpha \max(X, Y) < XY$  is

- (a)  $1/(2\alpha)$
- (b)  $\exp(1 - \alpha)$
- (c)  $1 - \alpha$
- (d)  $(1 - \alpha)^2$  ✓
- (e)  $1 - \alpha^2$

13. Imagine the first quadrant of the real plane as consisting of unit squares. A typical square has 4 corners:  $(i, j)$ ,  $(i + 1, j)$ ,  $(i + 1, j + 1)$ , and  $(i, j + 1)$ , where  $(i, j)$  is a pair of non-negative integers. Suppose a line segment  $\ell$  connecting  $(0, 0)$  to  $(90, 1100)$  is drawn. We say that  $\ell$  passes through a unit square if it passes through a point in the interior of the square. How many unit squares does  $\ell$  pass through?

- (a) 98,990
- (b) 9,900
- (c) 1,190
- (d) 1,180 ✓
- (e) 1,010

14. Consider the following  $3 \times 3$  matrices.

$$M_1 = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad M_2 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

How many 0-1 column vectors of the form

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

are there such that  $M_1 X = M_2 X$  (modulo 2)? (modulo 2 means all operations are done modulo 2, i.e.  $3 = 1$  (modulo 2),  $4 = 0$  (modulo 2)).

- (a) None
- (b) Two ✓
- (c) Three
- (d) Four
- (e) Eight

15. Let  $A$  and  $B$  be non-empty disjoint sets of real numbers. Suppose the the average of the numbers in the first set is  $\mu_A$  and the average of the numbers in the second set is  $\mu_B$ ; let the corresponding variances be  $v_A$  and  $v_B$  respectively. If the average of the of the elements in  $A \cup B$  is  $\mu = p \cdot \mu_A + (1 - p) \cdot \mu_B$ , what is the variance of the elements in  $A \cup B$ ?

- (a)  $p \cdot v_A + (1 - p) \cdot v_B$
- (b)  $(1 - p) \cdot v_A + p \cdot v_B$
- (c)  $p \cdot [v_A + (\mu_A - \mu)^2] + (1 - p) \cdot [v_B + (\mu_B - \mu)^2]$  ✓
- (d)  $(1 - p) \cdot [v_A + (\mu_A - \mu)^2] + p \cdot [v_B + (\mu_B - \mu)^2]$
- (e)  $p \cdot v_A + (1 - p) \cdot v_B + (\mu_A - \mu_B)^2$

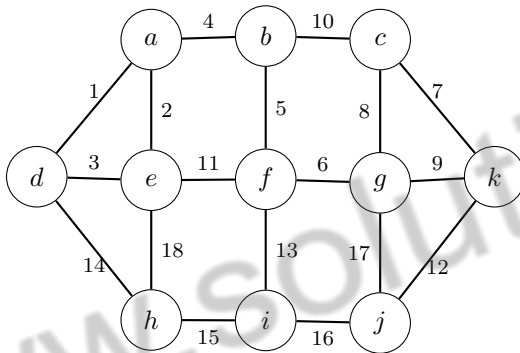
## Part B Computer Science

1. Consider the following recurrence relation:

$$T(n) = \begin{cases} 2T(\lfloor \sqrt{n} \rfloor) + \log n & \text{if } n \geq 2 \\ 1 & \text{if } n = 1. \end{cases}$$

Which of the following statements is TRUE?

- (a)  $T(n)$  is  $O(\log n)$ .
  - (b)  $T(n)$  is  $O(\log n \cdot \log \log n)$  but not  $O(\log n)$ . ✓
  - (c)  $T(n)$  is  $O(\log^{3/2} n)$  but not  $O(\log n \cdot \log \log n)$ .
  - (d)  $T(n)$  is  $O(\log^2 n)$  but not  $O(\log^{3/2} n)$ .
  - (e)  $T(n)$  is  $O(\log^2 n \cdot \log \log n)$  but not  $O(\log^2 n)$ .
2. Consider the following undirected connected graph  $G$  with weights on its edges as given in the figure below. A minimum spanning tree is a spanning tree of least weight and a maximum spanning tree is one with largest weight. A second-best minimum spanning tree is a spanning tree whose weight is the smallest among all spanning trees that are not minimum spanning trees in  $G$ .



Which of the following statements is TRUE in the above graph? (Note that all the edge weights are distinct in the above graph)

- (a) There is more than one minimum spanning tree and similarly, there is more than one maximum spanning tree here.
- (b) There is a unique minimum spanning tree, however there is more than one maximum spanning tree here.
- (c) There is more than one minimum spanning tree, however there is a unique maximum spanning tree here.
- (d) There is more than one minimum spanning tree and similarly, there is more than one second-best minimum spanning tree here.
- (e) There is a unique minimum spanning tree, however there is more than one second-best minimum spanning tree here. ✓

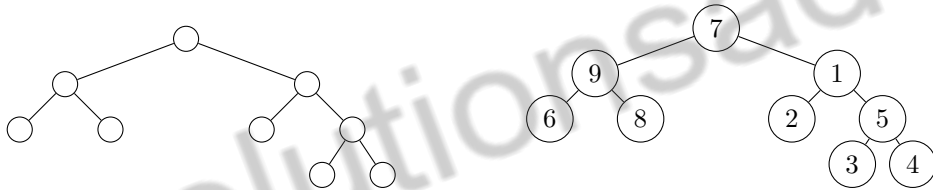
3. Consider the following code fragment in the C programming language when run on a non-negative integer  $n$ .

```
int f(int n)
{
    if(n==0 || n==1)
        return 1;
    else
        return f(n-1) + f(n-2);
}
```

Assuming a typical implementation of the language, what is the running time of this algorithm and how does it compare to the optimal running time for this problem?

- (a) This algorithm runs in polynomial time in  $n$  but the optimal running time is exponential in  $n$ .
- (b) This algorithm runs in exponential time in  $n$  and the optimal running time is exponential in  $n$ .
- (c) This algorithm runs in exponential time in  $n$  but the optimal running time is polynomial in  $n$ . ✓
- (d) This algorithm runs in polynomial time in  $n$  and the optimal running time is polynomial in  $n$ .
- (e) The algorithm does not terminate.

4. First, consider the tree on the left.



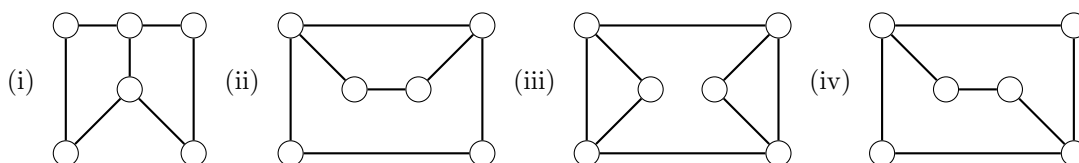
On the right, the nine nodes of the tree have been assigned numbers from the set  $\{1, 2, \dots, 9\}$  so that for every node, the numbers in its left subtree and right subtree lie in disjoint intervals (that is, all numbers in one subtree are less than all numbers in the other subtree). How many such assignments are possible? Hint: Fix a value for the root and ask what values can then appear in its left and right subtrees.

- (a)  $2^9 = 512$
- (b)  $2^4 \cdot 3^2 \cdot 5 \cdot 9 = 6480$  ✓
- (c)  $2^3 \cdot 3 \cdot 5 \cdot 9 = 1080$
- (d)  $2^4 = 16$
- (e)  $2^3 \cdot 3^3 = 216$

5. Suppose

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

is the adjacency matrix of an undirected graph with six vertices: that is, the rows and columns are indexed by vertices of the graph, and an entry is 1 if the corresponding vertices are connected by an edge and is 0 otherwise; the same order of vertices is used for the rows and columns. Which of the graphs below has the above adjacency matrix?



- (a) only (i)
- (b) only (ii)
- (c) only (iii)
- (d) only (iv)
- (e) (i) and (ii) ✓

6. Let  $B$  consist of all binary strings beginning with a 1 whose value when converted to decimal is divisible by 7.

- (a)  $B$  can be recognised by a deterministic finite state automaton. ✓
- (b)  $B$  can be recognised by a non-deterministic finite state automaton but not by a deterministic finite state automaton.
- (c)  $B$  can be recognised by a deterministic push-down automaton but not by a non-deterministic finite state automaton.
- (d)  $B$  can be recognised by a non-deterministic push-down automaton but not by a deterministic push-down automaton.
- (e)  $B$  cannot be recognised by any push down automaton, deterministic or non-deterministic.

7. Let  $a, b, c$  be regular expressions. Which of the following identities is correct ?

- (a)  $(a + b)^* = a^*b^*$
- (b)  $a(b + c) = ab + c$
- (c)  $(a + b)^* = a^* + b^*$
- (d)  $(ab + a)^*a = a(ba + a)^*$  ✓
- (e) None of the above



8. Let  $\Sigma_1 = \{a\}$  be a one letter alphabet and  $\Sigma_2 = \{a, b\}$  be a two letter alphabet. A language over an alphabet is a set of finite length words comprising letters of the alphabet. Let  $\mathcal{L}_1$  and  $\mathcal{L}_2$  be the set of languages over  $\Sigma_1$  and  $\Sigma_2$  respectively. Which of the following is true about  $\mathcal{L}_1$  and  $\mathcal{L}_2$ :

- (a) Both are finite.
- (b) Both are countably infinite.
- (c)  $\mathcal{L}_1$  is countable but  $\mathcal{L}_2$  is not.
- (d)  $\mathcal{L}_2$  is countable but  $\mathcal{L}_1$  is not.
- (e) Neither of them is countable. ✓

9. A Boolean expression is an expression made out of propositional letters (such as  $p, q, r$ ) and operators  $\wedge, \vee$  and  $\neg$ ; e.g.  $p \wedge \neg(q \vee \neg r)$ . An expression is said to be in *sum of product* form (also called *disjunctive normal form*) if all  $\neg$  occur just before letters and no  $\vee$  occurs in scope of  $\wedge$ ; e.g.  $(p \wedge \neg q) \vee (\neg p \wedge q)$ . The expression is said to be in *product of sum* form (also called *conjunctive normal form*) if all negations occur just before letters and no  $\wedge$  occurs in the scope of  $\vee$ ; e.g.  $(p \vee \neg q) \wedge (\neg p \vee q)$ . Which of the following is not correct?

- (a) Every Boolean expression is equivalent to an expression in sum of products form.
- (b) Every Boolean expression is equivalent to an expression in product of sum form.
- (c) Every Boolean expression is equivalent to an expression without  $\vee$  operator.
- (d) Every Boolean expression is equivalent to an expression without  $\wedge$  operator.
- (e) Every Boolean expression is equivalent to an expression without  $\neg$  operator. ✓

10. Consider the languages

$$\begin{aligned} L_1 &= \{a^m b^n c^p \mid (m = n \vee n = p) \wedge m + n + p \geq 10\} \\ L_2 &= \{a^m b^n c^p \mid (m = n \vee n = p) \wedge m + n + p \leq 10\} \end{aligned}$$

State which of the following is true?

- (a)  $L_1$  and  $L_2$  are both regular.
- (b) Neither  $L_1$  nor  $L_2$  is regular.
- (c)  $L_1$  is regular and  $L_2$  is not regular.
- (d)  $L_1$  is not regular and  $L_2$  is regular. ✓
- (e) Both  $L_1$  and  $L_2$  are infinite.

11. Let  $K_n$  be the complete graph on  $n$  vertices labelled  $\{1, 2, \dots, n\}$  with  $m = n(n-1)/2$  edges. What is the number of spanning trees of  $K_n$ ?

- (a)  $\binom{m}{n-1}$
- (b)  $m^{n-1}$
- (c)  $n^{n-2}$  ✓
- (d)  $n^{n-1}$
- (e) None of the above

12. Let  $t_n$  be the the sum of the first  $n$  natural numbers, for  $n > 0$ . A number is called triangular if it is equal to  $t_n$  for some  $n$ . Which of the following statements are true:

- (i) There exists three successive triangular numbers whose product is a perfect square.
- (ii) If the triangular number  $t_n$  is a perfect square, then so is  $t_{4n(n+1)}$ .
- (iii) The sum of the reciprocals of the first  $n$  triangular numbers is less than 2, i.e.

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \dots + \frac{1}{t_n} < 2.$$

- (a) (i) only
- (b) (ii) only
- (c) (iii) only
- (d) All of the above ✓
- (e) None of the above

13. Two undirected graphs  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  are said to be isomorphic if there exists a bijection  $\pi : V_1 \rightarrow V_2$  such that for all  $u, v \in V_1$ ,  $(u, v) \in E_1$  if and only if  $(\pi(u), \pi(v)) \in E_2$ . Consider the following language.

$$L = \{(G, H) | G \text{ and } H \text{ are undirected graphs such that a subgraph of } G \text{ is isomorphic to } H \}$$

Then, which of the following are true?

- (i)  $L \in NP$
  - (ii)  $L$  is NP-hard
  - (iii)  $L$  is undecidable
- (a) only (i)
  - (b) only (ii)
  - (c) only (iii)
  - (d) (i) and (ii) ✓
  - (e) (ii) and (iii)

14. Consider the following concurrent program (where statements separated by `||` within `cobegin-coend` are executed concurrently).

```
x:=1;
cobegin
  x:= x+1 || x:= x+1 || x:=x+1
coend
```

Reading and writing of variables is atomic but evaluation of expressions is not atomic. The set of possible values of  $x$  at the end of execution of the program is

- (a) {4}
- (b) {2, 3, 4} ✓
- (c) {2, 4}
- (d) {2, 3}
- (e) {2}

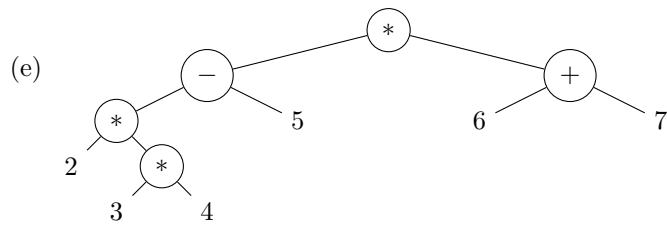
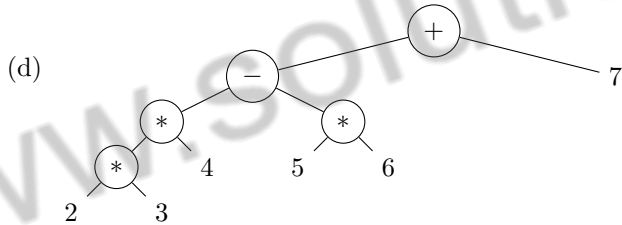
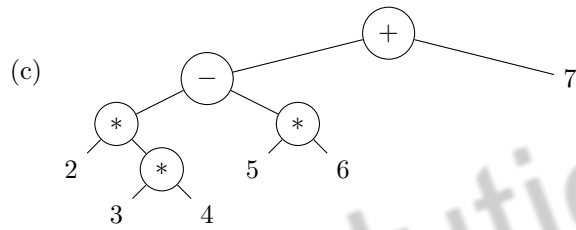
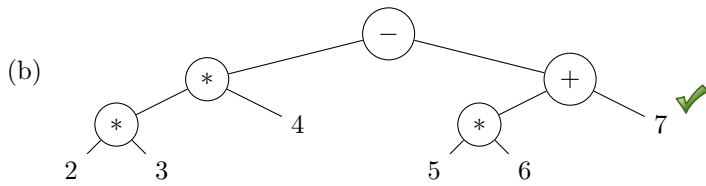
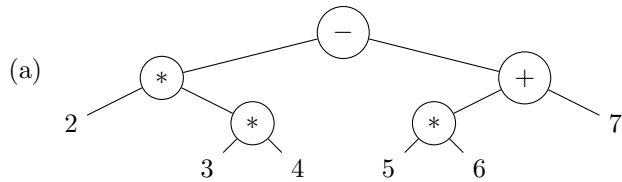
15. Consider the following grammar (the start symbol is  $E$ ) for generating expressions.

$$E \rightarrow T - E \mid T + E \mid T$$

$$T \rightarrow T * F \mid F$$

$$F \rightarrow 0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5 \mid 6 \mid 7 \mid 8 \mid 9$$

With respect to this grammar, which of the following trees is the valid evaluation tree for the expression  $2 * 3 * 4 - 5 * 6 + 7$ ?

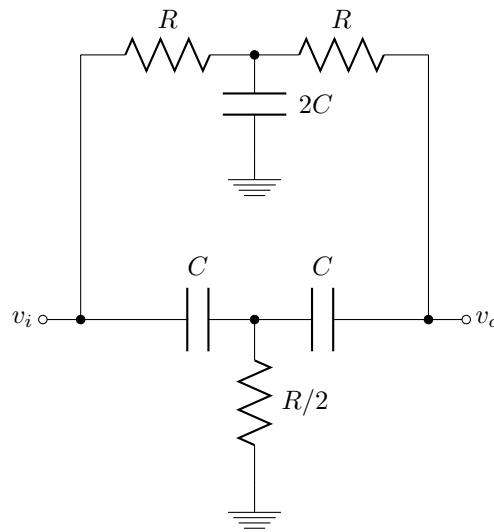


## Part C

### Systems Science

- For a time-invariant system, the impulse response completely describes the system if the system is
  - causal and non-linear
  - non-causal and non-linear
  - causal and linear ✓
  - All of the above
  - None of the above
- Let  $x[n] = a^{|n|}$ , ( $a$  is real,  $0 < a < 1$ ) and the discrete time Fourier transform (DTFT) of  $x[n]$  is given by  $X(\omega) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ . Then the DTFT of  $x[n]$  has which of the following properties?
  - It goes to zero at infinite number of values of  $\omega \in [-\pi, \pi]$
  - It goes to zero only at one value of  $\omega \in [-\pi, \pi]$
  - Its maximum value is larger than 1 ✓
  - Its minimum value is less than  $-1$
  - None of the above
- Let  $h(t)$  be the impulse response of an ideal low-pass filter with cut-off frequency 5kHz. Let  $g[n] = h(nT)$ , for integer  $n$ , be a sampled version of  $h(t)$  with sampling frequency  $\frac{1}{T} = 10\text{kHz}$ . The discrete-time filter with  $g[n]$  as its unit impulse response is a
  - low-pass filter
  - high-pass filter
  - band-pass filter
  - band-stop filter
  - all-pass filter ✓
- The capacity of a certain additive white Gaussian noise channel of bandwidth 1 MHz is known to be 8 Mbps when the average transmit power constraint is 50 mW. Which of the following statements can we make about the capacity  $C$  (in Mbps) of the same channel when the average transmit power is allowed to be 100 mW?
  - $C = 8$
  - $8 < C < 16$  ✓
  - $C = 16$
  - $C > 16$
  - There is not enough information to determine  $C$

5. What is the following passive circuit?



- (a) Low-pass filter
- (b) High-pass filter
- (c) Band-pass filter
- (d) Band-stop filter ✓
- (e) All-pass filter

6.  $\mathbf{A}$  is an  $n \times n$  square matrix of reals such that  $\mathbf{A}\mathbf{y} = \mathbf{A}^T\mathbf{y}$ , for all real vectors  $\mathbf{y}$ . Which of the following can we conclude?

- (i)  $\mathbf{A}$  is invertible
- (ii)  $\mathbf{A}^T = \mathbf{A}$
- (iii)  $\mathbf{A}^2 = \mathbf{A}$
- (a) Only (i)
- (b) Only (ii) ✓
- (c) Only (iii)
- (d) Only (i) and (ii)
- (e) None of the above

7. Let  $A$  be an  $8 \times 8$  matrix of the form

$$\begin{bmatrix} 2 & 1 & \dots & 1 \\ 1 & 2 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 2 \end{bmatrix}$$

Which of the following is true? [Hint: Use  $\det(I + AB) = \det(I + BA)$ ]

- (a)  $\det(A) = 9$  ✓
  - (b)  $\det(A) = 18$
  - (c)  $\det(A) = 14$
  - (d)  $\det(A) = 27$
  - (e) None of the above
8. Let  $X$  and  $Y$  be two independent and identically distributed random variables. Let  $Z = \max(X, Y)$  and  $W = \min(X, Y)$ . Which of the following is true?
- (a)  $Z$  and  $W$  are independent
  - (b)  $E(XZ) = E(YW)$
  - (c)  $E(XY) = E(ZW)$  ✓
  - (d) (a), (b), and (c)
  - (e) (a) and (b) only
9. Consider a random variable  $X$  that takes integer values 1 through 10 each with equal probability. Now consider random variable

$$Y = \min(7, \max(X, 4)).$$

What is the variance of  $Y$ ?

- (a)  $121/4$
  - (b)  $37/20$  ✓
  - (c)  $9/5$
  - (d)  $99/12$
  - (e) None of the above
10. Let  $X$  be a uniform random variable between  $[0, 1]$ . And let

$$M = \min_{m \geq 1, m \in \mathbb{N}} m.$$

Then which of the following is true?

- (a)  $E(M) = \infty$  ✓
- (b)  $E(M) \in [5, 10]$
- (c)  $E(M) = \exp(1)$
- (d)  $E(M) = \pi$
- (e) None of the above

11. For  $x > 0$ , for which range of values of  $\alpha$  is the following inequality true?

$$x \log_e(x) \geq x - \alpha$$

- (a)  $\alpha \geq 1/2$
- (b)  $\alpha \geq 0$
- (c)  $\alpha \leq 2$
- (d)  $\alpha \geq 1$  ✓
- (e) None of the above

12. Consider the following optimization problem

$$\max (2x + 3y)$$

subject to the following three constraints

$$\begin{aligned} x + y &\leq 5, \\ x + 2y &\leq 10, \text{ and} \\ x &< 3. \end{aligned}$$

Let  $z^*$  be the smallest number such that  $2x + 3y \leq z^*$  for all  $(x, y)$  which satisfy the above three constraints. Which of the following is true?

- (a) There is no  $(x, y)$  that satisfies the above three constraints.
- (b) All  $(x, y)$  that satisfy the above three constraints have  $2x + 3y$  strictly less than  $z^*$ .
- (c) There are exactly two  $(x, y)$  that satisfy the above three constraints such that  $2x + 3y$  equals  $z^*$ .
- (d) There is a unique  $(x, y)$  that satisfies the above three constraints such that  $2x + 3y$  equals  $z^*$ . ✓
- (e) There are infinitely many  $(x, y)$  that satisfy the above three constraints such that  $2x + 3y$  equals  $z^*$ .

13. Let

$$A = \begin{pmatrix} 1 & 1 + \varepsilon & 1 \\ 1 + \varepsilon & 1 & 1 + \varepsilon \\ 1 & 1 + \varepsilon & 1 \end{pmatrix}$$

Then for  $\varepsilon = 10^{-6}$ ,  $A$  has

- (a) only negative eigenvalues
- (b) only non-zero eigenvalues
- (c) only positive eigenvalues
- (d) one negative and one positive eigenvalue ✓
- (e) None of the above

14. Consider a frog that lives on two rocks  $A$  and  $B$  and moves from one rock to the other randomly. If it is at Rock  $A$  at any time, irrespective of which rocks it occupied in the past, it jumps back to Rock  $A$  with probability  $2/3$  and instead jumps to Rock  $B$  with probability  $1/3$ . Similarly, if it is at Rock  $B$  at any time, irrespective of which rocks it occupied in the past, it jumps back to Rock  $B$  with probability  $2/3$  and instead jumps to Rock  $A$  with probability  $1/3$ . After the first jump, it is at Rock  $A$ . What is the limiting probability that it is at Rock  $A$  after a total of  $n$  jumps as  $n \rightarrow \infty$ ?
- (a)  $1/2$  ✓
  - (b)  $2/3$
  - (c)  $1$
  - (d) The limit does not exist
  - (e) None of the above
15. Let  $x_1 = -1$  and  $x_2 = 1$  be two signals that are transmitted with equal probability. If signal  $x_i, i \in \{1, 2\}$  is transmitted, the received signal is  $y = x_i + n_i$ , where  $n_i$  is Gaussian distributed with mean  $\mu_i$  and variance  $\sigma_i$ . At the receiver, knowing  $y$ , your job is to detect whether  $x_1$  or  $x_2$  was sent. Let  $\theta$  be the detection threshold, i.e. if  $y < \theta$  then we declare  $x_1$  was transmitted, otherwise  $x_2$  was transmitted depending. In general, which of the following is true?
- (a) If  $\sigma_1 > \sigma_2$ , the optimal detection threshold  $\theta^*$  to minimize the probability of error is  $\leq 0$ .
  - (b) If  $\sigma_1 < \sigma_2$ , the optimal detection threshold  $\theta^*$  to minimize the probability of error is  $\leq 0$ .
  - (c) If  $\mu_1 > \mu_2$ , the optimal detection threshold  $\theta^*$  to minimize the probability of error is  $\leq 0$ .
  - (d) If  $\mu_1 < \mu_2$ , the optimal detection threshold  $\theta^*$  to minimize the probability of error is  $\leq 0$ .
  - (e) None of the above. ✓