

**GS-2018**  
**(Computer & Systems Sciences)**  
**TATA INSTITUTE OF FUNDAMENTAL RESEARCH**

Written Test in **COMPUTER & SYSTEMS SCIENCES - December 10, 2017**  
Duration : Three hours (3 hours)

Name : \_\_\_\_\_ Ref. Code : \_\_\_\_\_

**Please read all instructions carefully before you attempt the questions.**

1. Please fill-in details about name, reference code etc. on the answer sheet. The Answer Sheet is machine readable. Use only Black/Blue ball point pen to fill-in the answer sheet.
2. Indicate your ANSWER ON THE ANSWER SHEET by blackening the appropriate circle for each question. Do not mark more than one circle for any question : this will be treated as a wrong answer.
3. This question paper consists of three (3) parts. **Part-A** contains fifteen (15) questions and **must be attempted** by all candidates. **Part-B & Part-C** contain fifteen (15) questions each, directed towards candidates for (B) Computer Science and (C) Systems Science (including Communications and Applied Probability), respectively. **STUDENTS MAY ATTEMPT EITHER PART-B OR PART-C. In case, a student attempts both Parts B & C (no extra time will be given) and qualifies for interview in both B & C, he/she will have opportunity to be interviewed in both areas.** All questions carry equal marks. A correct answer for a question will give you +4 marks, a wrong answer will give you -1 mark, and a question not answered will not get you any marks.
4. We advise you to first mark the correct answers in the QUESTION SHEET and then to TRANSFER these to the ANSWER SHEET only when you are sure of your choice.
5. Rough work may be done on blank pages of the question paper. If needed, you may ask for extra rough sheets from an Invigilator.
6. **Use of calculators, mobile phones, laptops, tablets (or other electronic devices, including those connecting to the internet) is NOT permitted.**
7. Do NOT ask for clarifications from the invigilators regarding the questions. They have been instructed not to respond to any such inquiries from candidates. In case a correction/clarification is deemed necessary, the invigilator(s) will announce it publicly.

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## Part A: Common part

**Note:**

The questions in this part are to be answered by **all candidates**, i.e., **both Computer Science and Systems Science streams**.

1. Consider a point  $A$  inside a circle  $C$  that is at distance 9 from the centre of the circle. Suppose you are told that there is a chord of length 24 passing through  $A$  with  $A$  as its midpoint. How many distinct chords of  $C$  have integer length and pass through  $A$ ?
  - (a) 2
  - (b) 6
  - (c) 7
  - (d) 12 ✓
  - (e) 14

2. Consider the following subsets of  $\mathbb{R}^3$  (the first two are cylinders, the third is a plane):

$$C_1 = \{(x, y, z) : y^2 + z^2 \leq 1\};$$

$$C_2 = \{(x, y, z) : x^2 + z^2 \leq 1\};$$

$$H = \{(x, y, z) : z = 0.2\}.$$

Let  $A = C_1 \cap C_2 \cap H$ . Which of the following best describes the shape of the set  $A$ ?

- (a) Circle
  - (b) Ellipse
  - (c) Triangle
  - (d) Square ✓
  - (e) An octagonal convex figure with curved sides
3. Which of the following statements is TRUE for all sufficiently large integers  $n$ ?
    - (a)  $2^{2^{\sqrt{\log \log n}}} < 2^{\sqrt{\log n}} < n$  ✓
    - (b)  $2^{\sqrt{\log n}} < n < 2^{2^{\sqrt{\log \log n}}}$
    - (c)  $n < 2^{\sqrt{\log n}} < 2^{2^{\sqrt{\log \log n}}}$
    - (d)  $n < 2^{2^{\sqrt{\log \log n}}} < 2^{\sqrt{\log n}}$
    - (e)  $2^{\sqrt{\log n}} < 2^{2^{\sqrt{\log \log n}}} < n$

4. The distance from your home to your office is 4 kilometres and your normal walking speed is 4 km/hr. On the first day, you walk at your normal walking speed and take time  $T_1$  to reach office.

On the second day, you walk at a speed of 3 km/hr for 2 kilometres, and at a speed of 5 km/hr for the remaining 2 kilometres and you take time  $T_2$  to reach office.

On the third day, you walk at a speed of 3 km/hr for 30 minutes, and at 5 km/hr for the remaining time and take time  $T_3$  to reach office.

What can you say about the ordering of  $T_1$ ,  $T_2$  and  $T_3$ ?

- (a)  $T_1 > T_2$  and  $T_1 < T_3$ .
- (b)  $T_1 = T_2 = T_3$ .
- (c)  $T_1 < T_2$  and  $T_1 > T_3$ .
- (d)  $T_1 = T_2$  and  $T_1 < T_3$ .
- (e)  $T_1 < T_2$  and  $T_1 = T_3$ . ✓
5. Which of the following is the derivative of  $f(x) = x^x$  when  $x > 0$ ?
- (a)  $x^x$
- (b)  $x^x \ln x$
- (c)  $x^x + x^x \ln x$  ✓
- (d)  $(x^x)(x^x \ln x)$
- (e) None of the above; function is not differentiable for  $x > 0$
6. What is the minimum number of students needed in a class to guarantee that there are at least 6 students whose birthdays fall in the same month?
- (a) 6
- (b) 23
- (c) 61 ✓
- (d) 72
- (e) 91

7. Consider the following function definition.

```
void greet (int n)
{
    if (n>0)
    {
        printf("hello");
        greet(n-1);
    }
    printf("world");
}
```

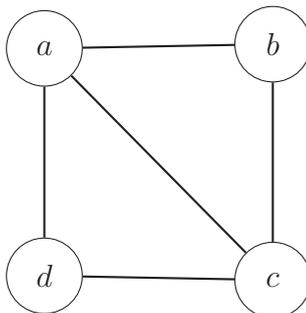
If you run `greet(n)` for some non-negative integer  $n$ , what would it print?

- (a)  $n$  times “hello”, followed by  $n + 1$  times “world” ✓
  - (b)  $n$  times “hello”, followed by  $n$  times “world”
  - (c)  $n$  times “helloworld”
  - (d)  $n + 1$  times “helloworld”
  - (e)  $n$  times “helloworld”, followed by “world”
8. A crime has been committed with four people at the scene of the crime. You are responsible for finding out who did it. You have recorded the following statements from the four witnesses, and you know one of them has committed the crime.
- (1) Anuj says that Binky did it.
  - (2) Binky says that Anuj did it.
  - (3) Chacko says that Binky is telling the truth.
  - (4) Desmond says that Chacko is not lying.

You know that exactly three of the statements recorded are FALSE. Who committed the crime?

- (a) Anuj
- (b) Binky ✓
- (c) Chacko
- (d) Desmond
- (e) Either Anuj or Binky; the information is insufficient to pinpoint the criminal

9. How many ways are there to assign colours from the range  $\{1, 2, \dots, r\}$  to the vertices of the following graph so that adjacent vertices receive distinct colours?



- (a)  $r^4$
- (b)  $r^4 - 4r^3$
- (c)  $r^4 - 5r^3 + 8r^2 - 4r$  ✓
- (d)  $r^4 - 4r^3 + 9r^2 - 3r$
- (e)  $r^4 - 5r^3 + 10r^2 - 15r$
10. Let  $C$  be a biased coin such that the probability of a head turning up is  $p$ . Let  $p_n$  denote the probability that an odd number of heads occurs after  $n$  tosses for  $n \in \{0, 1, 2, \dots\}$ . Then, which of the following is TRUE?
- (a)  $p_n = 1/2$  for all  $n \in \{0, 1, 2, \dots\}$ .
- (b)  $p_n = (1 - p)(1 - p_{n-1}) + p \cdot p_{n-1}$  for  $n \geq 1$  and  $p_0 = 0$ .
- (c)  $p_n = \sum_{i=1}^n p(1 - 2p)^{i-1}$  for  $n \geq 1$ . ✓
- (d) If  $p = 1/2$ , then  $p_n = 1/2$  for all  $n \in \{0, 1, 2, \dots\}$ .
- (e)  $p_n = 1$  if  $n$  is odd and 0 otherwise.

11. We are given a (possibly empty) set of objects. Each object in the set is colored either black or white; is shaped either circular or rectangular, and has a profile that is either fat or thin. These properties obey the following principles:
1. Each white object is also circular.
  2. Not all thin objects are black.
  3. Each rectangular object is also either thin or white or both thin and white.

Consider the following statements:

- (i) If there is a thin object in the set, then there is also a white object.
- (ii) If there is a rectangular object in the set, then there are at least two objects.
- (iii) Every fat object in the set is circular.

Which of the above statements must be TRUE for the set?

- (a) (i) only
  - (b) (i) and (ii) only
  - (c) (i) and (iii) only
  - (d) None of the statements must be TRUE
  - (e) All of the statements must be TRUE ✓
12. An  $n \times n$  matrix  $M$  with real entries is said to be *positive definite* if for every non-zero  $n$ -dimensional vector  $x$  with real entries, we have  $x^T M x > 0$ . Let  $A$  and  $B$  be symmetric, positive definite matrices of size  $n \times n$  with real entries. Consider the following matrices, where  $I$  denotes the  $n \times n$  identity matrix:

- (1)  $A + B$
- (2)  $ABA$
- (3)  $A^2 + I$

Which of the above matrices must be positive definite?

- (a) Only (2)
- (b) Only (3)
- (c) Only (1) and (3)
- (d) None of the above matrices are positive definite
- (e) All of the above matrices are positive definite ✓

13. A hacker knows that the password to the TIFR server is a 10-letter string consisting of lower-case letters from the English alphabet. He guesses a set of 5 distinct 10-letter strings (with lower-case letters) uniformly at random. What is the probability that one of the 5 guesses of the hacker is the correct password?
- (a)  $\frac{5}{(26)^{10}}$  ✓
- (b)  $1 - \left(1 - \frac{1}{(26)^{10}}\right)^5$
- (c)  $1 - \left(\frac{(26)^{10}-1}{(26)^{10}}\right) \left(\frac{(26)^{10}-2}{(26)^{10}}\right) \left(\frac{(26)^{10}-3}{(26)^{10}}\right) \left(\frac{(26)^{10}-4}{(26)^{10}}\right) \left(\frac{(26)^{10}-5}{(26)^{10}}\right)$
- (d)  $\frac{1}{(26)^{10}}$
- (e) None of the above
14. Let  $A$  be an  $n \times n$  invertible matrix with real entries whose row sums are all equal to  $c$ . Consider the following statements:
- (1) Every row in the matrix  $2A$  sums to  $2c$ .
- (2) Every row in the matrix  $A^2$  sums to  $c^2$ .
- (3) Every row in the matrix  $A^{-1}$  sums to  $c^{-1}$ .
- Which of the following is TRUE?
- (a) none of the statements (1), (2), (3) is correct
- (b) statement (1) is correct but not necessarily statements (2) or (3)
- (c) statement (2) is correct but not necessarily statements (1) or (3)
- (d) statements (1) and (2) are correct but not necessarily statement (3)
- (e) all the three statements (1), (2), and (3) are correct ✓
15. Suppose a box contains 20 balls: each ball has a distinct number in  $\{1, \dots, 20\}$  written on it. We pick 10 balls (without replacement) uniformly at random and throw them out of the box. Then we check if the ball with number “1” on it is present in the box. If it is present, then we throw it out of the box; else we pick a ball from the box uniformly at random and throw it out of the box.
- What is the probability that the ball with number “2” on it is present in the box?
- (a)  $9/20$
- (b)  $9/19$  ✓
- (c)  $1/2$
- (d)  $10/19$
- (e) None of the above

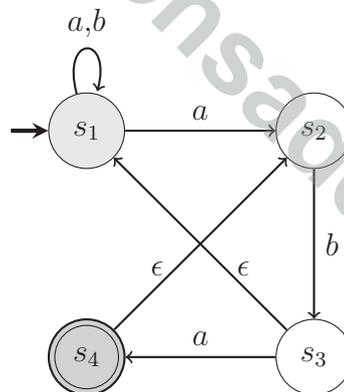
## Part B: Computer Science

### Note:

Only for Computer Science stream candidates.

If you are applying for the **Systems Science** stream, please skip this section and attempt questions from **Part C** on page 13.

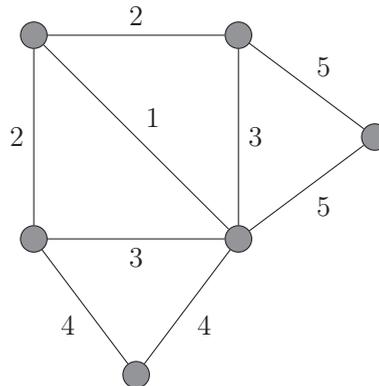
- What is the remainder when  $4444^{4444}$  is divided by 9?
  - 1
  - 2
  - 5
  - 7 ✓
  - 8
- Consider the following non-deterministic automaton, where  $s_1$  is the start state and  $s_4$  is the final (accepting) state. The alphabet is  $\{a, b\}$ . A transition with label  $\epsilon$  can be taken without consuming any symbol from the input.



Which of the following regular expressions corresponds to the language accepted by this automaton?

- $(a + b)^* aba$  ✓
- $aba(a + b)^* aba$
- $(a + b)aba(b + a)^*$
- $aba(a + b)^*$
- $(ab)^* aba$

3. How many distinct minimum weight spanning trees does the following undirected, weighted graph have?



- (a) 1  
 (b) 2  
 (c) 4 ✓  
 (d) 6  
 (e) 8
4. The notation “ $\Rightarrow$ ” denotes “implies” and “ $\wedge$ ” denotes “and” in the following formulae.  
 Let  $X$  denote the formula:  $(b \Rightarrow a) \Rightarrow (a \Rightarrow b)$   
 Let  $Y$  denote the formula:  $(a \Rightarrow b) \wedge b$
- Which of the following is TRUE?
- (a)  $X$  is satisfiable and  $Y$  is not satisfiable.  
 (b)  $X$  is satisfiable and  $Y$  is a tautology.  
 (c)  $X$  is not a tautology and  $Y$  is not satisfiable.  
 (d)  $X$  is not a tautology and  $Y$  is satisfiable. ✓  
 (e)  $X$  is a tautology and  $Y$  is satisfiable.
5. Which of the following functions, given by their recurrences, grows the fastest asymptotically?
- (a)  $T(n) = 4 \cdot T(n/2) + 10n$   
 (b)  $T(n) = 8 \cdot T(n/3) + 24n^2$   
 (c)  $T(n) = 16 \cdot T(n/4) + 10n^2$  ✓  
 (d)  $T(n) = 25 \cdot T(n/5) + 20(n \log n)^{1.99}$   
 (e) They are all asymptotically the same.

6. Consider the following implementation of a binary tree data structure. The operator + denotes list-concatenation. That is,  $[a,b,c] + [d,e] = [a,b,c,d,e]$ .

```
struct TreeNode:
    int value
    TreeNode leftChild
    TreeNode rightChild

function preOrder(T):
    if T == null:
        return []
    else:
        return [T.value] + preOrder(T.leftChild) + preOrder(T.rightChild)

function inOrder(T):
    if T == null:
        return []
    else:
        return inOrder(T.leftChild) + [T.value] + inOrder(T.rightChild)

function postOrder(T):
    if T == null:
        return []
    else:
        return postOrder(T.leftChild) + postOrder(T.rightChild) + [T.value]
```

For some T the functions inOrder(T) and preOrder(T) return the following:

```
inOrder(T):  [12,10,6,9,7,2,15,5,1,13,4,3,8,14,11]
preOrder(T): [5,2,10,12,9,6,7,15,13,1,3,4,14,8,11]
```

What does postOrder(T) return?

- (a) [12,6,10,7,15,2,9,1,4,13,8,11,14,3,5]
- (b) [11,8,14,4,3,1,13,15,7,6,9,12,10,2,5]
- (c) [11,14,8,3,4,13,1,5,15,2,7,9,6,10,12]
- (d) [12,6,7,9,10,15,2,1,4,8,11,14,3,13,5] ✓
- (e) Cannot be uniquely determined from given information.

7. Consider the recursive quicksort algorithm with “random pivoting”. That is, in each recursive call, a pivot is chosen uniformly at random from the sub-array being sorted. When this randomized algorithm is applied to an array of size  $n$  all whose elements are distinct, what is the probability that the smallest and the largest elements in the array are compared during a run of the algorithm?
- (a)  $\frac{1}{n}$
  - (b)  $\frac{2}{n}$  ✓
  - (c)  $\Theta\left(\frac{1}{n \log n}\right)$
  - (d)  $O(1/n^2)$
  - (e)  $\Theta\left(\frac{1}{n \log^2 n}\right)$
8. In an undirected graph  $G$  with  $n$  vertices, vertex 1 has degree 1, while each vertex  $2, \dots, n-1$  has degree 10 and the degree of vertex  $n$  is unknown. Which of the following statements must be TRUE of the graph  $G$ ?
- (a) There is a path from vertex 1 to vertex  $n$  ✓
  - (b) There is a path from vertex 1 to each vertex  $2, \dots, n-1$
  - (c) Vertex  $n$  has degree 1
  - (d) The diameter of the graph is at most  $n/10$
  - (e) All of the above choices must be TRUE
9. Let  $G = (V, E)$  be a DIRECTED graph, where each edge  $e$  has a positive weight  $w(e)$ , and all vertices can be reached from vertex  $s$ . For each vertex  $v$ , let  $\phi(v)$  be the length of the shortest path from  $s$  to  $v$ . Let  $G' = (V, E)$  be a new weighted graph with the same vertices and edges, but with the edge weight of every edge  $e = (u \rightarrow v)$  changed to  $w'(e) = w(e) + \phi(v) - \phi(u)$ .
- Let  $P$  be a path from  $s$  to a vertex  $v$ , and let  $w(P) = \sum_{e \in P} w_e$ , and  $w'(P) = \sum_{e \in P} w'_e$ . Which of the following options is NOT NECESSARILY TRUE?
- (a) If  $P$  is a shortest path in  $G$ , then  $P$  is a shortest path in  $G'$ .
  - (b) If  $P$  is a shortest path in  $G'$ , then  $P$  is a shortest path in  $G$ .
  - (c) If  $P$  is a shortest path in  $G$ , then  $w'(P) = 2 \times w(P)$ .
  - (d) If  $P$  is NOT a shortest path in  $G$ , then  $w'(P) < 2 \times w(P)$ .
  - (e) All of the above options are necessarily TRUE. ✓

10. For two  $n$ -bit strings  $x, y \in \{0, 1\}^n$ , define  $z := x \oplus y$  to be the bitwise XOR of the two strings (that is, if  $x_i, y_i, z_i$  denote the  $i$ -th bits of  $x, y, z$  respectively, then  $z_i = x_i + y_i \pmod{2}$ ). A function  $h : \{0, 1\}^n \rightarrow \{0, 1\}^n$  is called linear if  $h(x \oplus y) = h(x) \oplus h(y)$ , for every  $x, y \in \{0, 1\}^n$ . The number of such linear functions for  $n \geq 2$  is:

- (a)  $2^n$
- (b)  $2^{n^2}$  ✓
- (c)  $2^{\binom{n}{2}}$
- (d)  $2^{4n}$
- (e)  $2^{n^2+n}$

11. Consider the language  $L \subseteq \{a, b, c\}^*$  defined as

$$L = \{a^p b^q c^r : p = q \text{ or } q = r \text{ or } r = p\}.$$

Which of the following answers is TRUE about the complexity of this language?

- (a)  $L$  is regular but not context-free.
- (b)  $L$  is context-free but not regular. ✓
- (c)  $L$  is decidable but not context-free.
- (d) The complement of  $L$ , defined as  $\bar{L} = \{a, b, c\}^* \setminus L$ , is regular.
- (e)  $L$  is regular, context-free and decidable.

12. Consider the following statements:

- (i) For every positive integer  $n$ , let  $\#n$  be the product of all primes less than or equal to  $n$ . Then,  $\#p + 1$  is a prime, for every prime  $p$ .
- (ii)  $\pi$  is a universal constant with value  $22/7$ .
- (iii) No polynomial time algorithm exists that can find the greatest common divisor of two integers given as input in binary.
- (iv) Let  $L \equiv \{x \in \{0, 1\}^* \mid x \text{ is the binary encoding of an integer that is divisible by } 31\}$ . Then,  $L$  is a regular language.

Then which of the following is TRUE?

- (a) Only statement (i) is correct.
- (b) Only statement (ii) is correct.
- (c) Only statement (iii) is correct.
- (d) Only statement (iv) is correct. ✓
- (e) None of the statements are correct.

13. Let  $n \geq 3$ , and let  $G$  be a simple, connected, undirected graph with the *same* number  $n$  of vertices and edges. Each edge of  $G$  has a distinct real weight associated with it. Let  $T$  be the minimum weight spanning tree of  $G$ . Which of the following statements is NOT ALWAYS TRUE?
- The minimum weight edge of  $G$  is in  $T$ .
  - The maximum weight edge of  $G$  is not in  $T$ . ✓
  - $G$  has a unique cycle  $C$  and the minimum weight edge of  $C$  is also in  $T$ .
  - $G$  has a unique cycle  $C$  and the maximum weight edge of  $C$  is not in  $T$ .
  - $T$  can be found in  $O(n)$  time from the adjacency list representation of  $G$ .
14. Define the language  $\text{INFINITE}_{\text{DFA}} \equiv \{\langle A \rangle \mid A \text{ is a DFA and } L(A) \text{ is an infinite language}\}$ , where  $\langle A \rangle$  denotes the description of the deterministic finite automata (DFA). Then which of the following about  $\text{INFINITE}_{\text{DFA}}$  is TRUE:
- It is regular.
  - It is context-free but not regular.
  - It is Turing decidable (recursive). ✓
  - It is Turing recognizable but not decidable.
  - Its complement is Turing recognizable but it is not decidable.
15.  $G$  represents an undirected graph and a cycle refers to a simple cycle (no repeated edges or vertices). Define the following two languages.

$$\begin{aligned} \text{SCYCLE} &= \{\langle G, k \rangle \mid G \text{ contains a cycle of length at most } k\} \\ &\text{and} \\ \text{LCYCLE} &= \{\langle G, k \rangle \mid G \text{ contains a cycle of length at least } k\} \end{aligned}$$

Which of the following is NOT known to be TRUE (to the best of our current knowledge)?

- $\text{SCYCLE} \in \text{P}$ .
- $\text{LCYCLE} \in \text{NP}$ .
- $\text{LCYCLE} \leq_P \text{SCYCLE}$  (i.e, there is a polynomial time many-to-one reduction from  $\text{LCYCLE}$  to  $\text{SCYCLE}$ ) ✓
- $\text{LCYCLE}$  is NP-complete.
- $\text{SCYCLE} \leq_P \text{LCYCLE}$  (i.e, there is a polynomial time many-to-one reduction from  $\text{SCYCLE}$  to  $\text{LCYCLE}$ )

—END OF CS SECTION—

(SS SECTION BEGINS IN THE FOLLOWING PAGE)

## Part C: Systems Science

**Note:**

Only for Systems Science stream candidates.

1. Consider a discrete-time system which in response to input sequence  $x[n]$  ( $n$  integer) outputs the sequence  $y[n]$  such that

$$y[n] = \begin{cases} 0, & n = -1, -2, -3, \dots, \\ \frac{1}{2}(y[n-1] + nx[n]), & n = 0, 1, 2, \dots \end{cases}$$

Which of the following describes the system?

- (a) Linear, time-invariant
- (b) Linear, time-variant ✓
- (c) Non-linear, time-invariant
- (d) Non-linear, time-variant
- (e) Cannot be determined from the information given
2. A hotel has  $n$  rooms numbered  $1, 2, \dots, n$ . For each room there is one spare key labeled with the room number. The hotel manager keeps all the spare keys in a box. Her mischievous son got hold of the box and permuted the labels uniformly at random. What is the expected number of keys which still open the room whose label they carry? [Hint: Use linearity of expectation]
- (a) 1 ✓
- (b)  $\frac{n-1}{n}$
- (c)  $\frac{n}{n-1}$
- (d)  $\frac{n}{2}$
- (e) None of the above
3. Let  $\lim_{n \rightarrow \infty} f(n) = \infty$  and  $\lim_{n \rightarrow \infty} g(n) = \infty$ . Then which of the following is necessarily TRUE.
- (a)  $\lim_{n \rightarrow \infty} |f(n) - g(n)| = \infty$
- (b)  $\lim_{n \rightarrow \infty} |f(n) - g(n)| = 0$
- (c)  $\lim_{n \rightarrow \infty} |f(n)/g(n)| = \infty$
- (d)  $\lim_{n \rightarrow \infty} |f(n)/g(n)| = 1$
- (e) None of the above ✓

4. Consider

$$f(x) = \frac{(x \log x + x)^5 (1 + 2/x)^x}{(x + 1/x)^5 (\log x + 1/\log x)^6}.$$

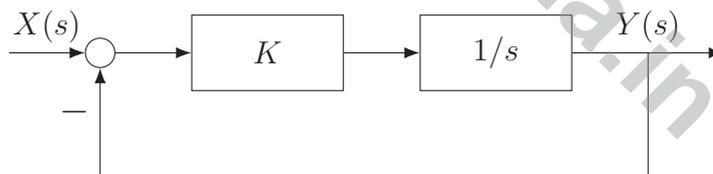
What can we say about  $\lim_{x \rightarrow \infty} f(x)$ ?

- (a) The function  $f(x)$  does not have a limit as  $x \rightarrow \infty$
  - (b)  $\lim_{x \rightarrow \infty} f(x) = e^2$
  - (c)  $\lim_{x \rightarrow \infty} f(x) = e^{1/2}$
  - (d)  $\lim_{x \rightarrow \infty} f(x) = 0$  ✓
  - (e)  $\lim_{x \rightarrow \infty} f(x) = \infty$
5. Suppose  $\vec{u}, \vec{v}_1, \vec{v}_2 \in \mathbb{R}^n$  are linearly independent vectors. Let the pair of real numbers  $(a_1^*, a_2^*)$  be such that they solve the following optimization problem

$$d = \min_{a_1, a_2 \in \mathbb{R}} \|\vec{u} - (a_1 \vec{v}_1 + a_2 \vec{v}_2)\|,$$

where for a vector  $\vec{w} \in \mathbb{R}^n$  we denote its length by  $\|\vec{w}\|$ . Let  $\vec{v}_* = a_1^* \vec{v}_1 + a_2^* \vec{v}_2$ , so that  $d = \|\vec{u} - \vec{v}_*\|$ . Which of the following is equal to  $d^2$ ?

- (a)  $\|\vec{v}_1\|^2 + \|\vec{v}_2\|^2 - \|\vec{u}\|^2$
  - (b)  $\|\vec{u}\|^2 - \|\vec{v}_1\|^2 - \|\vec{v}_2\|^2$
  - (c)  $\|\vec{u}\|^2 - \|\vec{v}_*\|^2$  ✓
  - (d)  $\|\vec{v}_*\|^2 - \|\vec{u}\|^2$
  - (e) None of the above
6. Consider the system shown below.



If  $K > 0$ , which of the following describes the system?

- (a) Stable, causal ✓
- (b) Stable, non-causal
- (c) Unstable, non-causal
- (d) Unstable, causal
- (e) Cannot be determined from the information given

7. Let  $X_1, X_2$  and  $X_3$  be independent random variables with uniform distribution over  $[0, \theta]$ . Consider the following statements.

- (i)  $E[\max\{X_1, X_2, X_3\}] = \frac{3}{4}\theta$ .
- (ii)  $E[\max\{X_1, X_2\}] - E[\max\{X_2, X_3\}] = 0$ .
- (iii)  $E[X_1] = \theta/2$
- (iv)  $E[\max\{X_1, X_2\}] = \frac{2}{3}\theta$

Which of the above statements is/are TRUE?

- (a) Only (i)
  - (b) Only (ii)
  - (c) Only (iii)
  - (d) Only (iv)
  - (e) All of (i) – (iv) ✓
8. Let  $A$  be an  $n \times n$  real matrix for which two distinct non-zero  $n$ -dimensional real column vectors  $v_1, v_2$  satisfy the relation  $Av_1 = Av_2$ . Consider the following statements.

- (i) At least one eigenvalue of  $A$  is zero.
- (ii)  $A$  is not full rank.
- (iii) Columns of  $A$  are not linearly independent.
- (iv)  $\det(A) = 0$ .

Which of the above statements is/are TRUE?

- (a) Only (i)
- (b) Only (ii)
- (c) Only (iii)
- (d) Only (iv)
- (e) All of (i) – (iv) ✓

9. Let  $X$  and  $Y$  be two independent and identically distributed binary random variables that take values  $\{-1, +1\}$  each with probability  $1/2$ . Let  $Z_1 = \max(X, Y)$ , and  $Z_2 = \min(X, Y)$ . Consider the following statements.

- (i)  $Z_1$  and  $Z_2$  are uncorrelated
- (ii)  $Z_1$  and  $Z_2$  are independent
- (iii)  $P(Z_1 = Z_2) = \frac{1}{2}$

Which of the above statements is/are TRUE?

- (a) Only (i)
  - (b) Only (ii)
  - (c) Only (iii) ✓
  - (d) Both (i) and (ii), but not (iii)
  - (e) All of (i), (ii) and (iii)
10. Suppose that  $X_1$  and  $X_2$  denote the random outcomes of independent rolls of two dice each of which takes six values 1, 2, 3, 4, 5, 6 with equal probability. What is the conditional expectation

$$E[X_1 | \max(X_1, X_2) = 5]$$

- (a) 3
  - (b) 4
  - (c)  $35/9$  ✓
  - (d)  $5/2$
  - (e)  $15/4$
11. Assume the following well known result: If a coin is flipped independently many times and its probability of heads ( $H$ ) is  $p \in (0, 1)$  and probability of tails ( $T$ ) is  $(1 - p)$ , then the expected number of coin flips till the first time a heads is observed is  $1/p$ .

What is the expected number of coin flips till the sequence  $HT$ , i.e., tails immediately following a heads, is observed for the first time?

- (a)  $\frac{1}{1-(1-p)^2} \left( 2 + \frac{1+p^2}{1-p} + p \right)$
- (b)  $\frac{2}{p(1-p)}$
- (c)  $2 + \frac{1}{p(1-p)}$
- (d)  $\frac{1}{1-(1-p)^2} (4 + 1/p)$
- (e)  $\frac{1}{p} + \frac{1}{1-p}$  ✓

12. Suppose that Amitabh Bachchan has ten coins in his pocket. 3 coins have tails on both sides. 4 coins have heads on both sides. 3 coins have heads on one side and tails on the other and both the outcomes are equally likely when that coin is flipped. In a bet with Dharmendra, Amitabh picks up a coin at random (each coin is equally likely to be picked) from these ten coins, flips it and finds that the outcome is tails. What is the probability that the other side of this coin is heads?
- (a)  $1/2$
  - (b)  $3/10$
  - (c)  $1/4$
  - (d)  $0.3$
  - (e)  $1/3$  ✓

13. Consider five distinct binary vectors  $X_1, \dots, X_5$  each of length 10. Let

$$d_{ij} = \sum_{k=1}^{10} (X_{ik} \text{ XOR } X_{jk}),$$

(i.e.,  $d_{ij}$  is the number of coordinates where  $X_i$  and  $X_j$  differ) be the Hamming distance between  $X_i$  and  $X_j$  and let  $d = \min_{i,j=1,\dots,5,i \neq j} d_{ij}$ . Which of the following is TRUE? [Hint: Look at the first two entries of  $X_1$  to  $X_5$ , and argue about the result noting that there are five binary vectors.]

- (a)  $d = 10$
  - (b)  $d = 9$
  - (c)  $d = 8$
  - (d)  $d < 8$  ✓
  - (e) Information is not sufficient
14. Define the  $\ell_p$  ball in two dimensions as the set of points  $(x, y)$  such that  $|x|^p + |y|^p \leq 1$ . Which of the following is FALSE:
- (a) The  $\ell_2$  ball is contained in the  $\ell_3$  ball
  - (b) The  $\ell_2$  ball is contained in the  $\ell_1$  ball ✓
  - (c) The  $\ell_3$  ball is contained in the  $\ell_4$  ball
  - (d) The  $\ell_2$  ball is contained in the  $\ell_5$  ball
  - (e) The  $\ell_1$  ball is contained in the  $\ell_3$  ball

15. Consider real-valued continuous functions  $f : [0, 2] \rightarrow (-\infty, \infty)$  and let

$$A = \int_0^1 |f(x)| dx \quad \text{and} \quad B = \int_1^2 |f(x)| dx.$$

Which of the following is TRUE?

(a) There exists an  $f$  so that

$$A + B < \int_0^2 f(x) dx$$

(b) There exists a strictly negative  $f$ , that is  $f(x) < 0$  for all  $x \in [0, 2]$ , such that

$$\int_0^2 f(x) dx = A + B = B - A$$

(c) There exists such an  $f$  so that

$$\int_0^2 f(x) dx = A + B = A - B \quad \checkmark$$

(d) There does not exist an  $f$  such that  $\int_0^1 f(x) dx = 3$

(e) There does not exist an  $f$  so that

$$A + B \leq - \int_0^2 f(x) dx$$

— END OF PAPER —