CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 26 May 2011

This question paper has 6 printed sides. Part A has 10 questions of 3 marks each. Part B has 7 questions of 10 marks each. The total marks are 100.

Part A

1. Let $L \subseteq \{0,1\}^*$ be a language accepted by a finite automaton. Let F be some subset of $\{0,1\}^*$, containing 2011 strings. Which of the following statements is true? Justify your answer.

   (a) $L \cup F$ is always regular.

   (b) $L \cup F$ is regular only when $L \cap F = \emptyset$—that is, when L and F do not have any common string.

   (c) $L \cup F$ is never regular.

   (d) $L \cup F$ is regular only if L contains ϵ (the empty string).

2. You have two six-sided cubic dice but they are numbered in a strange manner. On the first die, two opposite faces are numbered 1, two opposite faces are numbered 3 and the last pair of opposite faces are numbered 6. On the second die, the three pairs of opposing faces are numbered 2, 4 and 5. Both dice are fair: each side has an equal probability of coming face up when tossed.

   Which of the following statements is *not* true of this pair of unusual dice?

   (a) The probability that the sum of the values shown by the dice is 5 is the same as probability that the sum is 8.

   (b) The probability that the sum is odd is higher than the probability that the sum is even.

   (c) The probability that the sum is strictly less than 7 is the same as the probability that the sum is strictly greater than 7.

   (d) The probability that the sum is a multiple of 5 is the same as the probability that the sum is a prime number.

3. You have a bag with 347 black balls and 278 white balls. Without looking, you pick up two balls from the bag and apply the following rule.

   - If both balls are of the same colour, you throw them both away.
   - Otherwise, you throw away the black ball and return the white ball to the bag.

   You keep repeating this process. If at some stage there is exactly one ball left in the bag, which of the following is true?

   (a) The ball in the bag is definitely white.

(b) The ball in the bag is definitely black.

(c) Both colours are possible, but the probability of it being white is greater.

(d) Both colours are possible, but the probability of it being black is greater.

4. Lavanya has to complete 12 courses for her degree. There are six compulsory courses: Basic and Advanced Mathematics, Basic and Advanced Physics and Basic and Advanced Electronics. She also has to complete six Optional Courses. Each course takes one semester to complete. There are some constraints because of prerequisites.

   - For Mathematics, Physics and Electronics, the Basic course must be completed before starting the Advanced course.
   - Advanced Physics must be completed before starting Basic Electronics.
   - Advanced Mathematics must be completed before starting Advanced Physics.
   - The Optional Courses can be done in any order, but no more than two Optional Courses can be taken in a semester.

   Given these constraints, what is the minimum number of semesters that Lavanya needs to complete her course requirements.

   (a) 3 (b) 4 (c) 5 (d) 6

5. A 3-ary boolean function is a function that takes three boolean arguments and produces a boolean output. Let f and g be 3-ary boolean functions. We say that f and g are *neighbours* if f and g agree on at least one input and disagree on at least one input: that is, there exist x, y, z such that $f(x, y, z) = g(x, y, z)$ and x', y', z' such that $f(x', y', z') \neq g(x', y', z')$.

   Suppose we fix a 3-ary boolean function h. How many neighbours does h have?

   (a) 128 (b) 132 (c) 254 (d) 256

6. A valuable sword belonging to the Grand King was stolen, and the three suspects were Ibn, Hasan, and Abu. Ibn claimed that Hasan stole it, and Hasan claimed that Abu stole it. It was not clear that one of them stole it, but it was later learnt that no innocent person had lied. It was also learnt that the sword was stolen by only one person.

   Who stole the sword?

   (a) Ibn (b) Hasan (c) Abu (d) None of them

7. Let $G = (V, E)$ be a graph. Define $\overline{G}$ to be $(V, \overline{E})$, where for all $u, v \in V$, $(u, v) \in \overline{E}$ if and only if $(u, v) \notin E$. Then, which of the following is true:

   (a) $\overline{G}$ is always connected.

   (b) $\overline{G}$ is connected iff G is not connected.

   (c) At least one of G and $\overline{G}$ is connected.

   (d) G is not connected or $\overline{G}$ is not connected.

8. In programming languages like C, C++, Python ... the memory used by a program is typically separated into two parts, the *stack* and the *heap*. Consider the following statements:

   (A) A stack is efficient for managing nested function calls.

   (B) Stack space is limited while heap space is not.

   (C) The stack cannot be used for persistent data structures.

   Then:

   (a) A and B are true but C is false.

   (b) A and C are true but B is false.

   (c) B and C are true but A is false.

   (d) All three statements are true.

9. You have a laptop with a fixed amount of memory and hard disk space and no external storage devices connected (CD, USB drives, ...). Which of the following is the most accurate formal model of your laptop?

   (a) Turing machine

   (b) Linear bounded automaton

   (c) Pushdown automaton

   (d) Finite state automaton

10. Consider the following functions `f` and `g`

```
f(){                    g(){
    x = x+1;                y = y+1;
    x = y*y;                y = x*x;
    x = x-y;                y = y-x;
}                       }
```

   Suppose we start with initial values of `1` for `x` and `2` for `y` and then execute `f` and `g` in parallel—that is, at each step we either execute one statement from `f` or one statement from `g`. Which of the following is not a possible final state?

   (a) `x = 2, y = 2` (b) `x = 5, y = -1` (c) `x = -63, y = 72` (d) `x = 32, y = 5`

Part B

1. A multinational company is developing an industrial area with many buildings. They want to connect the buildings with a set of roads so that:

   - Each road connects exactly two buildings.
   - Any two buildings are connected via a sequence of roads.

- Omitting any road leads to at least two buildings not being connected by any sequence of roads..

(i) Is it always possible to colour each building with either red or blue so that every road connects a red and blue building?

Two roads are said to be *adjacent* to each other if they serve a common building. A set of roads is said to be *preferred* if:

- No two roads in the set are adjacent, and,
- Each building is served by at least one road in the set.

(ii) Is it always possible to find a preferred set of roads?

(iii) Is it ever possible to find two sets of preferred roads differing in at least one road?

Substantiate your answers by either proving the assertion or providing a counterexample.

2. Let G be a connected graph. For a vertex x of G we denote by $G - x$ the graph formed by removing x and all edges incident on x from G. G is said to be good if there are at least two distinct vertices x, y in G such that both $G - x$ and $G - y$ are connected.

   (i) Show that for any subgraph H of G, H is good if and only if G is good.

   (ii) Given a good graph, devise a linear time algorithm to find two such vertices.

   (iii) Show that there exists a graph H such that we cannot find three distinct vertices u_1, u_2, u_3 such that each of $G - u_1$, $G - u_2$, and $G - u_3$ is connected.

3. Your team is playing a chess tournament against a visiting team. Your opponents have arrived with a team of M players, numbered $1, 2, \ldots, M$. You have N players, numbered $1, 2, \ldots, N$ from which to choose your team, where $N \geq M$.

   Each of the M players from the visiting team must be paired up with one of your N players. The tournament rules insist that the pairings must respect the order that has been fixed for both teams. That is, when you pick players $i_1, i_2, \ldots i_M$, to play against opponents numbered $1, 2, \ldots, M$, it must be the case that $i_1 < i_2 < \ldots < i_M$, in terms of the order $1, 2, \ldots, N$ in which your players are listed.

   You want to ensure a good fight, so you plan to pick your team so that the teams are as evenly balanced as possible. Each player j on your team has a numerical score $YS(j)$ that represents his or her playing ability. Likewise, each player i in the opponent team has a playing ability indicated by a numerical score $OS(i)$. The difference in strength between a player i_j from your team and his or her opponent j on the visiting team is the absolute value $|YS(i_j) - OS(j)|$. The imbalance of a pairing is the sum of these differences across all M match-ups in the pairing. Your aim is to minimize this imbalance.

   For instance suppose you have six players, whose strengths are as follows.

i	1	2	3	4	5	6
$YS(i)$	2	3	4	1	5	7

Also, suppose that the visiting team has three players, whose strengths are as follows.

i	1	2	3
$OS(i)$	2	9	2

In this situation, the most balanced pairing is (1,1), (3,2) and (4,3), which yields an imbalance of
$|YS(1) - OS(1)| + |YS(3) - OS(2)| + |YS(4) - OS(3)| = |2 - 2| + |4 - 9| + |1 - 2| = 6$

Propose an efficient algorithm to solve this problem and analyse its complexity.

4. Let $L \subseteq \{0,1\}^*$. Suppose L is regular and there is a nondeterministic automaton N which recognizes L. Define the reverse of the language L to be the language $L^R = \{w \in \{0,1\}^* \mid \mathit{reverse}(w) \in L\}$ – here $\mathit{reverse}(w)$ denotes the string w read in reverse. For example $\mathit{reverse}(0001) = 1000$.

   (i) Show that L^R is regular. How can you use N to construct an automata to recognize L^R?

   (ii) Show that the language $L \cdot L^R \triangleq \left\{x \in \{0,1\}^* \mid \exists y, z \text{ where } x = yz, y \in L, z \in L^R\right\}$ is regular. How can you use N to construct an automata for $L \cdot L^R$?

5. Let $G = (V, E)$ be a undirected graph. We say $S \subseteq V$ is a clique if and only if for all $u, v \in S$, the edge (u, v) is in E.

   Now, let $G = (V, E)$ be a graph in which each vertex has degree at most 5. Give an algorithm to find the largest clique in G. What is the complexity of your algorithm?

6. Consider a plate stacked with several disks, each of a different diameter (they could all be, for instance, *dosas* or *chapatis* of different sizes). We want to sort these disks in decreasing order according to their diameter so that the widest disk is at the bottom of the pile. The only operation available for manipulating the disks is to pick up a stack of them from the top of the pile and invert that stack. (This corresponds to lifting up a stack *dosas* or *chapatis* between two big spoons and flipping the stack.)

   (i) Give an algorithm for sorting the disks using this operation.

   (ii) How many steps will your algorithm take in the worst case?

7. A finite sequence of bits is represented as a list with values from the set `{0,1}`—for example, `[0,1,0]`, `[1,0,1,1]`, ...

   `[]` denotes the empty list, and `[b]` is the list consisting of one bit `b`. For a nonempty list `l`, `head(l)` returns the first element of `l`, and `tail(l)` returns the list obtained by removing the first element from `l`. `a:l` denotes a new list formed by adding a at the head of list l.

   For example:

   - `head([0,1,0]) = 0`, `tail([0,1,0]) = [1,0]`,
   - `head([1]) = 1`, `tail([1]) = []`, and
   - `1:[0,1,0] = [1,0,1,0]`.

Consider the following functions:

- `xor` takes takes as input two bits and returns a bit.

```
xor(a,b)
  if (a == b) return(0)
  else return(1)
  endif
```

- `f1` takes as input a list and returns another list.

```
f1(s)
  if (s == []) then return([1])
  else if (head(s) == 0) then return(1:tail(s))
  else if (head(s) == 1) then return(0:f1(tail(s)))
  endif
```

- `f2` takes as input a bit and a list and returns a bit.

```
f2(b,s)
  if (s == []) then return(b)
  else if (head(s) == 0) then return(f2(xor(b,1),tail(s)))
  else if (head(s) == 1) then return(xor(b,1))
  endif
```

- `g1` takes as input a nonnegative number and returns a list.

```
g1(n)
  if (n == 0) then return([0])
  else return f1(g1(n-1))
  endif
```

- `g2` takes as input a nonnegative number and returns a bit.

```
g2(n)
  if (n == 0) then return(0)
  else return f2(g2(n-1),g1(n))
  endif
```

(i) What is the value of `g2(7)` and `g2(8)`?

(ii) What is the value of `g2(256)` and `g2(257)`?