DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

PhD Qualifier Examination, Paper I

Total time: 2 Hours **October 28, 2009** **Maximum Marks: 120**

[Answer ALL the parts]

Part A: Discrete mathematics

[Answer ALL questions]

A.1 Let Σ be the alphabet $\{a, b, c\}$. Show that the number of words of length n in which the letter a appears an even number of times is $(3^n + 1)/2$. (You may use induction or any other proof technique.) **[10]**

A.2 For any positive integer n, it is possible to draw n circles in the Euclidean plane such that every pair of circles intersects at two distinct points, and no three circles have a point in common. Let a_n denote the total number of regions formed by n such intersecting circles.

(a) Compute a_1, a_2, a_3 and a_4. **[1]**

(b) Find a recurrence relation satisfied by a_n for $n \geq 1$. **[4]**

(c) Solve the recurrence relation above. **[5]**

A.3 (a) How many onto (surjective) functions are there from a set with six elements to a set with three elements? **[5]**

(b) Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is injective and f is surjective, then g is injective. **[5]**

A.4 A cabinet has 3 drawers. In the first drawer, there are 2 gold balls, in the second drawer, there are two silver balls, and in the third drawer, there are one gold ball and one silver ball. A drawer is picked at random, and a ball is chosen at random from the two balls in the drawer. Given that a gold ball was drawn, what is the probability that the drawer with the two gold balls was chosen? **[10]**

Part B: Algorithms

[Answer ANY FOUR questions]

B.1 The running time of an algorithm is $T(n)$ for an input of size n. Suppose that $T(n)$ satisfies the following recurrence relation.

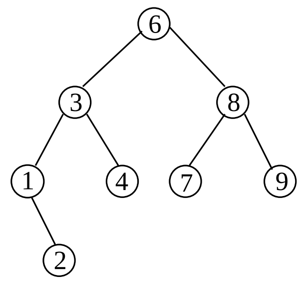
$$\begin{aligned} T(1) &= 1, \\ T(n) &= 3T(n-1) + 1 \text{ for } n \geq 2. \end{aligned}$$

Prove that $T(n) = \Theta(3^n)$. **[10]**

B.2 Explain how the merge sort algorithm sorts the following array of eight elements. **[10]**

7	3	8	2	1	9	6	4

B.3 Consider the following binary search tree.



(a) Suppose that the new value 5 is to be inserted in this tree. Indicate which nodes of the tree are to be visited for locating the insertion position. Draw the tree after this insertion. **[3+2]**

(b) After this insertion, the value 3 is to be deleted from the tree. Explain which nodes in the tree are to be visited for the deletion. Draw the tree after this deletion. **[3+2]**

B.4 Explain an efficient algorithm to detect the presence of cycles in an undirected graph. What is the running time of your algorithm? **[7+3]**

B.5 Let LONG-CYCLE denote the computational problem of deciding whether a given undirected graph G on n vertices contains a cycle of length $\geq n/2$. Prove that LONG-CYCLE is NP-complete. You may assume that the Hamiltonian cycle problem (on undirected graphs) is NP-complete. **[10]**

Part C: Formal Languages and Automata Theory

[Answer ANY FOUR questions]

C.1 Design a DFA for the language **[10]**

$$L = \{w \mid w \text{ contains an equal number of occurrences of } 01 \text{ and } 10 \text{ as substrings}\}.$$

(Note that the strings 101 and 010 are in L.)

C.2 Let $L = \{w \mid w$ contains an even number of a's and an odd number of b's and does not contain the substring $ab\}$. Give a DFA that recognizes L and a regular expression that generates L. (*Hint*: Describe L more simply.) **[8+2]**

C.3 (a) Indicate with proper justification whether the following statement is *true* or *false*: If N is an NFA that recognizes the language L, swapping the accept and nonaccept states yields a new NFA that recognizes the complement of L. **[2]**

(b) Give a recognizer for the language $L = \{a^i b^j c^k \mid i = j \text{ or } j = k\}$. Explain the working of the recognizer. **[8]**

C.4 (a) Let G be the grammar

$S \rightarrow aB \mid bA,$
$A \rightarrow a \mid aS \mid bAA,$
$B \rightarrow b \mid bS \mid aBB.$

Is the grammar unambiguous? Explain your answer. **[5]**

(b) Consider the following decision problem: Given any CFG G, is $\mathcal{L}(G) = \emptyset$? Show that it is a decidable problem by giving a decision algorithm for the problem. **[5]**

C.5 Give a context-free grammar for the set of all strings over the alphabet $\{a, b\}$ with exactly twice as many a's as b's. Explain the working of the grammar by characterizing the strings generated by each non-terminal. **[10]**
