DEPARTMENT OF COMPUTER SCIENCE & ENGINEERING

PhD Comprehensive Test, Paper I

Total time: 3 Hours **October 04, 2010** **Maximum Marks: 120**

Answer ALL the parts

Part A: Discrete Mathematics

Answer ALL questions

A.1 Let (x_i, y_i), $i = 1, 2, 3, 4, 5$, be a set of 5 distinct points with integer coordinates in the x-y plane. Show that the midpoint of the line joining at least one pair of these points has integer coordinates. **(6)**

A.2 For each of the following relations, state whether they fulfill each of the 4 main properties: reflexive, symmetric, antisymmetric, transitive. Briefly substantiate each of your answers.

(a) The coprime relation on Z. (Recall that $a, b \in Z$ are coprime if and only if $\gcd(a, b) = 1$.) **(4)**

(b) Divisibility on Z. **(4)**

A.3 (a) Find a recurrence relation for the number of bit strings of length n that contain two consecutive 0's. **(5)**

(b) What are the initial conditions? **(2)**

(c) How many bit strings of length 7 contain two consecutive 0s? **(4)**

A.4 Suppose we roll a die until a 6 comes up, at which time we stop rolling.

(a) What is the probability we roll the die n times? **(5)**

(b) What is the expected number of times we roll the die? **(5)**

A.5 What is the largest possible number of vertices in a graph with 35 edges where every vertex has degree at least 3? **(5)**

Part B: Algorithms

Answer ALL questions

B.1 Suppose that you need to sort a list of n integers. To this effect, you break the list into t sub-lists each of size (about) n/t. You then sort each sub-list using bubble sort, and finally merge the t sorted sub-lists to a single sorted list.

(a) Describe how you can use a min-heap to merge the t sorted sub-lists efficiently. **(5)**

(b) Deduce the running time of this algorithm (in terms of n and t) **(5)**

B.2 Let $G = (V, E)$ be a digraph, and v a specific vertex in V. Assume that the edges of G are labeled by positive costs.

(a) Describe a method by which you can compute the shortest distance from each $u \in V \setminus \{v\}$ to v. (We can call this problem the *single-destination-shortest-path problem*.) **(6)**

(b) What is the running time of your algorithm? **(2)**

B.3 You are given an $n \times n$ grid with a positive value associated with every segment of the grid. The grid may be assumed to be located in a treasure island, and the value may be assumed to be the amount of treasure buried in that segment. You are supposed to move from the lower left corner to the upper right corner of the grid. Each step of your movement involves taking a rightward segment or an upward segment. While traversing a segment, you collect the treasure that you find on the segment. Your task is to maximize the treasure collected.

(a) Write a dynamic programming algorithm to solve this maximization problem. **(8)**

(b) Analyze the time complexity of your algorithm. **(4)**

B.4 Let $G = (V, E)$ be an undirected graph. A k-clique in G is a subset $V' \subseteq V$ with $|V'| = k$ and with $(u, v) \in E$ for all pairs u, v of vertices in V'. A k-independent set in G is a subset $V' \subseteq V$ with $|V'| = k$ and with $(u, v) \notin E$ for all pairs u, v of vertices in V'.

(a) Given that the computational problem of deciding whether a given graph G contains a k-clique (for a given k) is NP-complete, prove that the computational problem of deciding whether a given graph G contains a k-independent set (for a given k) is NP-complete too. **(4)**

(b) Let $G = (V, E)$ with $|V| = n$. Devise a polynomial-time algorithm to decide whether G contains an $(n - 1)$-independent set. **(4)**

(c) Does the result of Part (b) prove that $\mathrm{P} = \mathrm{NP}$? Justify. **(2)**

Part C: Formal Languages and Automata Theory

Answer ALL questions

C.1 Construct a DFA equivalent to the NFA shown below: **(10)**

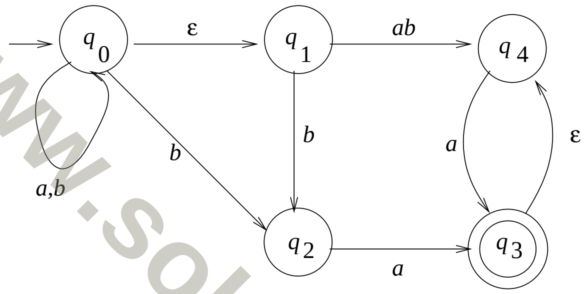

Figure 1: NFA

C.2 Let F be the language of all strings over $\{0, 1\}$ that do not contain a pair of 1's that are separated by an odd number of symbols. Give the state diagram of a DFA with five states that recognizes F. **(6)**

C.3 Consider the CFG $G = \langle V, \Sigma, R, S \rangle$, where

$$V = \{S, A, B\}, \Sigma = \{a, b\}, R = \{S \to aB \,|\, bA, A \to a \,|\, aS \,|\, bAA, B \to b \,|\, bS \,|\, aBB\}.$$

Prove that $L(G)$ is the set of all strings in $\{a, b\}^+$ that have equal number of occurrences of a and b. (*Hint*: Form three conjectures characterizing the strings produced by the variables S, A, B and prove them.) **(12)**

C.4 Consider the language $L = \{a^m b^n \mid n \leq m \leq 2n\}$ over $\Sigma = \{a, b\}$.

(a) Give a recognizer for L. Give only algorithmic description of the recognizer. **(5)**

(b) Give a CFG G for L. Justify that $\mathcal{L}(G) = L$. **(7)**
