CHENNAI MATHEMATICAL INSTITUTE

M.Sc. / Ph.D. Programme in Computer Science

Entrance Examination, 2022

Part A has 10 questions of 3 marks each. Each question in Part A has four choices, of which exactly one is correct. Part B has 7 questions of 10 marks each. The total marks are 100. Answers to Part A must be entered directly on the computer.

Part A

1. If Vinay finishes his homework and the school closes early, then he can play in the park or eat an ice cream. He will end up at the dispensary with tummy ache if he eats ice cream and plays in the park. Which of the following can be correctly inferred?

   (a) If he doesn't end up in the dispensary with tummy ache, then he did not finish his homework or the school closed late.

   (b) If he doesn't end up in the dispensary with tummy ache, he didn't eat ice cream and he didn't play in the park.

   (c) Both (a) and (b)

   (d) None of the above.

2. There are n members of Chennai Mathematical Institute. Most of them are very studious, and like to own lots of books. Now the following facts have been learnt.

   - No two members own exactly the same number of books.
   - Each member owns strictly less than n books.
   - No member has exactly 200 books.

   Given the above information, which of the following is not a possible value of n?

   (a) 100

   (b) 199

   (c) 200

   (d) 201

3. Which of the following assertions about regular languages is *incorrect*?

   (a) Every subset of a regular language is regular.

   (b) For every regular language L, there is a subset of L that is regular.

   (c) For every language L, there is a superset of L that is regular.

   (d) The complement of every regular language is regular.

4. Consider the following languages over the alphabet $\{a, b, c, d\}$

   - $L_1 = \{a^n b^n c^m d^m \mid n, m \geq 0\}$
   - $L_2 = \{a^n b^m c^n d^m \mid n, m \geq 0\}$
   - $L_3 = \{a^n b^m c^m d^n \mid n, m \geq 0\}$

   Which of these languages is/are context-free?

   (a) None of them.
   (b) Only L_1 and L_2.
   (c) Only L_1 and L_3.
   (d) All of them.

5. As part of a class activity, students in a class of 50 were asked to keep track of the total number of hours that they spent looking at the screen of some digital device on a specific day. It was found that the average screentime for the class was 4 hours. What is the maximum possible number of students with at least 16 hours of screentime?

   (a) 11
   (b) 12
   (c) 13
   (d) 14

6. The Telvio mobile service provider allows each customer to choose a part of their 10-digit mobile number when getting a new connection. The first two digits of the number are fixed by the company based on the customer's region. The customer can choose the *last* four digits as they wish. The company chooses each of the remaining four digits uniformly at random, and without replacement, from the list $\{0, 1, 2, \ldots, 9\}$. Note that this means that the digits in positions $3, 4, 5$ and 6 in a Telvio number are all different.

   What is the probability that in the mobile number assigned to a new customer by Telvio, the digits in positions $3, 4, 5$ and 6 appear in increasing order when read from left to right?

   (a) $\frac{1}{4}$
   (b) $\frac{1}{16}$
   (c) $\frac{1}{24}$
   (d) $\frac{1}{32}$

7. Consider a random graph G on n vertices where for each pair of vertices u, v, there is an edge (u, v) with probability $p \in [0, 1]$. What is the expected number of cycles of length 3 in this graph?

   (a) $\binom{n}{3} \cdot p^3$
   (b) $\frac{p^3}{\binom{n}{3}}$
   (c) $n^3 p^3$
   (d) None of the above

Questions 8 and 9 refer to the following two functions.

```
int f(int m) {
    int a, b, c, d;
    a = 0; b = 0;
    c = 0; d = 1;
    while (a < m) {
        a = a + 1;
        b = b + c;
        int temp = d;
        d = c;
        c = temp;
    }
    return b;
}
```

```
int g(int m) {
    int a = 1;
    int i = 0;
    while (i < m) {
        i = i+1;
        a = 2*a;
    }
    return a;
}
```

8. What is the result of `f(100)`?

   (a) 100
   (b) 5050
   (c) 50
   (d) 1

9. If `g(f(n)) = 32`, which of the following is a possible value of `n`?

   (a) 8
   (b) 11
   (c) 5
   (d) 64

10. What can you conclude from the following statements about problems A and B?

    (I) There is a polynomial-time algorithm to solve A.
    (II) There is an exponential-time algorithm to solve B.
    (III) B can be reduced to A in polynomial-time.

    (a) Not all of them can be simultaneously true.
    (b) There is a polynomial-time algorithm for B.
    (c) A cannot be reduced to B in polynomial-time.
    (d) There is no exponential-time algorithm for A.

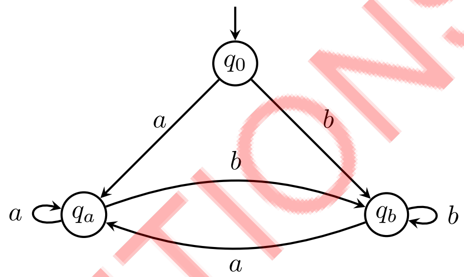
Part B

1. A *Muller automaton* is defined as a tuple $M = (Q, I, \Sigma, \rightarrow, T)$ where:

   - Q is a finite set of *states*;
   - $I \subseteq Q$ is the set of *initial states*;
   - Σ is the finite *alphabet*;
   - $\rightarrow \subseteq Q \times \Sigma \times Q$ is the *transition relation*; and
   - $T \subseteq 2^Q$ is the *accept table*.

   A run of M on a word $x = a_1 \dots a_n$ is a sequence of the form $\rho = q_0 a_1 q_1 \cdots q_{n-1} a_n q_n$ where $q_0 \in I$ and $q_{i-1} \xrightarrow{a_i} q_i$ for each $i \leq n$. For a run ρ as above, we define $vs(\rho) = \{q_0, \dots, q_n\}$, the set of *visited states* along the run ρ. We say that M accepts a word x if there is a run ρ of M on x such that $vs(\rho) \in T$. The language accepted by M, denoted $L(M)$, is the set $\{x \in \Sigma^* \mid x \text{ is accepted by } M\}$.

   Consider the Muller automaton whose states and transitions are depicted below. The initial state is $\{q_0\}$.



   (a) Consider the run $\rho_1 = q_0 a q_a a q_a b q_b$ on the word aab. What is $vs(\rho_1)$? Now, consider the run $\rho_2 = q_0 b q_b b q_b$ on the word bb. What is $vs(\rho_2)$?

   (b) What is the language accepted by the above automaton when the accept table is $\{\{q_0, q_a, q_b\}\}$?

   (c) What should the accept table be in order to accept a^*?

2. Consider the language L over the alphabet $\{a, b\}$ given below.

   $$L = \{w \mid w \text{ has equal number of } a\text{'s and } b\text{'s, and there are no adjacent } a\text{'s.}\}$$

   For instance, the words $abba, abab$ are in the language but not bab and $baab$.

   (a) Prove that L does not contain any word that starts and ends with a b.

   (b) Give a context-free grammar for L.

3. We say that an integer a is co-prime to another integer b if $\gcd(a, b) = 1$. For any integer n, $\varphi(n)$ is the number of integers from 1 up to $|n|$ that are co-prime to n.

   (a) Calculate $\varphi(5)$, $\varphi(10)$ and $\varphi(20)$.

   (b) Show that $\varphi(p) = p - 1$ for any prime p.

   (c) Prove that if a is co-prime to b then the remainder of a when divided by b is also co-prime to b.

4. You are organizing a party involving $2n$ diplomats. Each pair of diplomats are either friends or enemies. You have managed to invite an excellent set of guests, each of whom has more friends than enemies (among the other guests). Can you now seat them at a round table so that everyone has two friends as their neighbours. (**Hint:** Model this situation as an appropriate graph so that the desired seating arrangement is a Hamiltonian path.)

5. For any set S of natural numbers, we say that a relation $R \subseteq S \times S$ is a 2-spanner of S if it satisfies the following conditions:

   - $(i, j) \in R \Rightarrow i < j$;
   - $i < j \Rightarrow [((i, j) \in R) \text{ or } (\exists k : (i, k) \in R \wedge (k, j) \in R)]$.

   For example, $\{(1, 2), (2, 3)\}$ is a 2-spanner for $\{1, 2, 3\}$, and

   $$R_1 = \{(1, 2), (1, 4), (2, 3), (2, 4), (3, 4), (4, 5), (4, 6), (4, 7), (5, 6), (6, 7)\}$$

   is a 2-spanner for $S = \{1, \ldots, 7\}$. There are other 2-spanners for S, of course. $R_2 = \{(i, j) \mid i, j \in \{1, \ldots, 7\}, i < j\}$ is an example. But R_1 is of size 10, while R_2 is of size 21. We would like to find 2-spanners that are as small as possible.

   (a) Suppose you are given 2-spanners R_1 and R_2 for $\{1, \ldots, 7\}$ and $\{9, \ldots, 15\}$ respectively, each of size 10. Use them to construct a 2-spanner R for $\{1, \ldots, 15\}$. Try to get R of size 34.

   (b) Generalize the above construction to show that any set S of size $2^k - 1$ (for $k > 2$) has a 2-spanner of size $(k - 2)2^k + 2$.

6. There is a treasure hunt game in which parts of a treasure are hidden across n islands, numbered 1 to n. You start in island 1, and aim to collect all parts of the treasure, by hopping from island to island. Each island has a list of other islands you can directly go to. (Note that if you can directly go from island i to island j, it does not necessary mean that you can directly go from j to i.) You can revisit the same island multiple times during your search. Design an algorithm that takes as input the number n of islands, the n neighbourhood lists, and determines if you can succeed in collecting all parts of the treasure. The algorithm should run in time $O(n^2 \cdot 2^n)$.

7. Consider the following inventory problem. You are running a company that sells lorries. Predictions tell you the quantity of sales to expect over the next n months. Let d_i denote the number of sales expected in month i. We assume that sales happen on the first of the month, and that lorries not sold are stored till the start of the next month. You can store at most C lorries, and it costs R to store each lorry for a month. You receive lorries from the manufacturer in shipments, each of which has a transportation fee of F (regardless of the number of lorries ordered). You start out with no lorries. Your aim is to place orders (say l_i is the number of lorries ordered in month i) so as to satisfy the following constraints:

   - For each i, the number of lorries on hand at the start of month i (l_i+ whatever is stored from the previous month) is enough to meet the demand d_i.
   - For each i, the number of lorries left over after meeting the demand d_i should not exceed the storage capacity C.

   The aim is to determine the orders $(l_1, \ldots, l_n)$ that will minimize the overall transportation fee and the overall storage cost.

   For example, if $n = 4$, and the demands for each month is given by $10, 11, 8, 12$, and if $F = 50$, while $R = 2$ and $C = 10$, then here are a few possible scenarios.

Month	1	2	3	4	Total
d_i	10	11	8	12	
l_i	10	11	8	12	
Cost	50	50	50	50	200
l_i	20	11	0	10	
Cost	$50 + 10 \times 2$	$50 + 10 \times 2$	2×2	50	194
l_i	10	19	0	12	
Cost	50	$50 + 8 \times 2$	0	50	166

   Let $c_i(S)$ denote the minimal cost of transportation and storage to meet demands from month i till month n, given that we have S lorries left over at the start of month i, while satisfying the storage requirements.

   (a) Write an expression for $c_n(S)$.

   (b) Express $c_i(S)$ in terms of $c_{i+1}(S')$ for appropriate values of S'.

   (c) Convert the above equations into a dynamic programming algorithm that computes $c_1(0)$. Your algorithm must run in time polynomial in n and C.