Part A: Common part

Note:
The questions in this part are to be answered by **all candidates**, i.e., **both Computer Science and Systems Science streams.**

1. A snail crawls up a vertical pole 75 feet high, starting from the ground. Each day it crawls up 5 feet, and each night it slides down 4 feet. When will it first reach the top of the pole?

(a) 75^{th} day

(b) 74^{th} day

(c) 73^{rd} day

(d) 72^{nd} day

(e) 71^{st} day ✓

2. We would like to invite a minimum number n of people (their birthdays are independent of each other) to a party such that the expected number of pairs of people that share the same birthday is at least 1. What should n be?

(Ignore leap years, so there are only 365 possible birthdays. Assume that birthdays fall with equal probability on each of the 365 days of the year.)

(a) 23

(b) 28 ✓

(c) 92

(d) 183

(e) 366

3. A binary string is a sequence of 0's and 1's. A binary string is *finite* if the sequence is finite, otherwise it is *infinite*. Examples of finite binary strings include 00010100, and 1111101010. Which of the following is TRUE about the set of all finite binary strings and the set of all infinite binary strings?

(a) The set of all finite binary strings is countable while the set of all infinite binary strings is uncountable ✓

(b) The set of all finite binary strings is uncountable while the set of all infinite binary strings is countable

(c) The set of all finite binary strings and the set of all infinite binary strings are both countable

(d) The set of all finite binary strings and the set of all infinite binary strings are both uncountable

(e) The set of all finite binary strings is countable while whether the set of all infinite binary strings is countable or not is not known

4. Consider the polynomial $p(x) = x^3 - x^2 + x - 1$. How many *symmetric* matrices with integer entries are there whose characteristic polynomial is p? (Recall that the *characteristic polynomial* of a square matrix A in the variable x is defined to be the determinant of the matrix $(A - xI)$ where I is the identity matrix.)

(a) 0 ✓

(b) 1

(c) 2

(d) 4

(e) Infinitely many

5. Let $\mathcal{F}$ be the set of all functions mapping $\{1, \ldots, n\}$ to $\{1, \ldots, m\}$. Let f be a function that is chosen uniformly at random from $\mathcal{F}$. Let x, y be distinct elements from the set $\{1, \ldots, n\}$. Let p denote the probability that $f(x) = f(y)$. Then,

(a) $p = 0$

(b) $p = \frac{1}{n^m}$

(c) $0 < p \leq \frac{1}{m^n}$

(d) $p = \frac{1}{m}$ ✓

(e) $p = \frac{1}{n}$

6. Let f be a polynomial of degree $n \geq 3$ all of whose roots are non-positive real numbers. Suppose that $f(1) = 1$. What is the maximum possible value of $f'(1)$?

(a) 1

(b) n ✓

(c) $n + 1$

(d) $\frac{n(n+1)}{2}$

(e) $f'(1)$ can be arbitrarily large given only the constraints in the question

7. Initially, N white beads are arranged in a circle. A number k is chosen uniformly at random from $\{1, \ldots, N - 1\}$. Then a set of k beads is chosen uniformly from the white beads, and these k beads are coloured black. The position of the beads remains unchanged. What is the probability that the black beads occur sequentially in the circle, i.e., at most two black beads have white beads next to them?

(a) $\dfrac{2N}{2N+1}$

(b) $\dfrac{N^2}{(N-1)\,(N-1)!}$

(c) $\dfrac{N}{N-1} \displaystyle\sum_{k=1}^{N-1} \frac{1}{\binom{N}{k}}$ ✓

(d) $\dfrac{1}{N} + \displaystyle\sum_{k=1}^{N-1} \frac{1}{\binom{N}{k}}$

(e) None of the above

8. Let A be the $(n+1) \times (n+1)$ matrix given below, where $n \geq 1$. For $i \leq n$, the i-th row of A has every entry equal to $2i - 1$ and the last row, i.e., the $(n+1)$-th row of A has every entry equal to $-n^2$.

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 3 & 3 & \cdots & 3 \\ \vdots & \vdots & \vdots & \vdots \\ 2n-1 & 2n-1 & \cdots & 2n-1 \\ -n^2 & -n^2 & \cdots & -n^2 \end{bmatrix}$$

Which of the following statements is TRUE for all $n \geq 1$?

(a) A has rank n

(b) A^2 has rank 1

(c) All the eigenvalues of A are distinct

(d) All the eigenvalues of A are 0 ✓

(e) None of the above

9. You are given the following properties of sets A, B, X, and Y. For notation, $|A|$ denotes the cardinality of set A (i.e., the number of elements in A), and $A \setminus B$ denotes the set of elements that are in A but not in B.

1. $A \cup B = X \cup Y$
2. $A \cap B = X \cap Y = \emptyset$
3. $|Y \setminus A| = 2$
4. $|A \setminus X| = 4$

Which of the following statements MUST then be FALSE?

(a) $|X| = 5$

(b) $|Y| = 5$ ✓

(c) $|A \cup X| = |B \cup Y|$

(d) $|A \cap X| = |B \cap Y|$

(e) $|A| = |B|$

10. Consider a bag containing colored marbles. There are n marbles in the bag such that there is exactly one pair of marbles of color i for each $i \in \{1, \ldots, m\}$ and the rest of the marbles are of distinct colors (different from colors $\{1, \ldots, m\}$). You draw two marbles uniformly at random (without replacement). What is the probability that both marbles are of same color?

(a) $\frac{m}{n}$

(b) $\frac{2m}{n}$

(c) $\frac{2m}{n(n-1)}$ ✓

(d) $\frac{2m}{n^2}$

(e) $\frac{m}{n(n-1)}$

11. Let X be a finite set. A family $\mathcal{F}$ of subsets of X is said to be *upward closed* if the following holds for all sets $A, B \subseteq X$:

$$A \in \mathcal{F} \text{ and } A \subseteq B \Rightarrow B \in \mathcal{F}.$$

For families $\mathcal{F}$ and $\mathcal{G}$ of subsets of X, let

$$\mathcal{F} \sqcup \mathcal{G} = \{A \cup B : A \in \mathcal{F} \text{ and } B \in \mathcal{G}\}.$$

Suppose $\mathcal{F}$ and $\mathcal{G}$ are upward closed families. Then which of the following is true?

(a) $\mathcal{F} \sqcup \mathcal{G} = \mathcal{F} \cap \mathcal{G}$ ✓

(b) $\mathcal{F} \sqcup \mathcal{G} = \mathcal{F} \cup \mathcal{G}$

(c) $\mathcal{F} \sqcup \mathcal{G} = \mathcal{F} \setminus \mathcal{G}$

(d) $\mathcal{F} \sqcup \mathcal{G} = \mathcal{G} \setminus \mathcal{F}$

(e) None of the above

12. Alice plays the following game on a math show. There are 7 boxes and identical prizes are hidden inside 3 of the boxes. Alice is asked to choose a box where a prize might be. She chooses a box uniformly at random. From the unchosen boxes which do not have a prize, the host opens an arbitrary box and shows Alice that there is no prize in it. The host then allows Alice to change her choice if she so wishes. Alice chooses a box uniformly at random from the other 5 boxes (other than the one she chose first and the one opened by the host). Her probability of winning the prize is

(a) 3/7

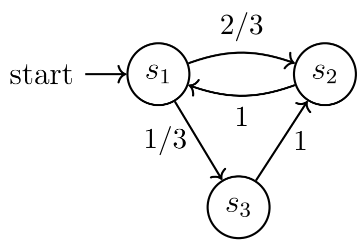
(b) 1/2

(c) 17/30

(d) 18/35 ✓

(e) 9/19

13. Consider the transition system shown in the figure below with the initial state s_1. A token is initially placed at s_1, and it moves to s_2 with probability $\frac{2}{3}$, and to s_3 with probability $\frac{1}{3}$. From s_2 and s_3, the token always moves to s_1 and s_2 respectively. A *run* of the system consists of an infinite sequence of states constructed by moving the token from one state to another following the transitions forever. Assuming such a run is chosen randomly, what is the fraction of times that the state s_2 is expected to appear in the run?



(a) $\frac{1}{7}$

(b) $\frac{2}{7}$

(c) $\frac{3}{7}$ ✓

(d) $\frac{5}{7}$

(e) None of the above

14. Suppose $w(t) = 4e^{it}$, $x(t) = 3e^{i(t+\pi/3)}$, $y(t) = 3e^{i(t-\pi/3)}$ and $z(t) = 3e^{i(t+\pi)}$ are points that move in the complex plane as the time t varies in $(-\infty, \infty)$. Let $c(t)$ be the point in the complex plane such that $|w(t)-c(t)|^2+|x(t)-c(t)|^2+|y(t)-c(t)|^2+|z(t)-c(t)|^2$ is minimum. For each value of t, the point $c(t)$ is unique, but $c(t)$ moves at constant speed as t varies. At what speed? That is, what is $\left|\frac{\mathrm{d}}{\mathrm{d}t}c(t)\right|$?

(a) $\frac{1}{2\pi}$

(b) 2π

(c) $\sqrt{3}\pi$

(d) $\frac{1}{\sqrt{3}\pi}$

(e) 1 ✓

15. Fix $n \geq 4$. Suppose there is a particle that moves randomly on the number line, but never leaves the set $\{1, 2, \ldots, n\}$. The initial probability distribution of the particle is π i.e., the probability that particle is in location i is given by $\pi(i)$. In the first step, if the particle is at position i, it moves to one of the positions in $\{1, 2, \ldots, i\}$ with uniform distribution; in the second step, if the particle is in location j, then it moves to one of the locations in $\{j, j+1, \ldots, n\}$ with uniform distribution. Suppose after two steps, the final distribution of the particle is uniform. What is the initial distribution π?

(a) π is not unique

(b) π is uniform

(c) $\pi(i)$ is non-zero for all even i and zero otherwise

(d) $\pi(1) = 1$ **and** $\pi(i) = 0$ **for** $i \neq 1$ ✓

(e) $\pi(n) = 1$ and $\pi(i) = 0$ for $i \neq n$

—End of Common Section—

(CS section begins in the following page)

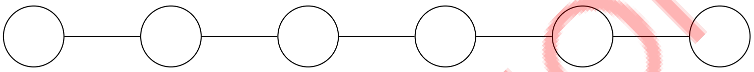
Part B: Computer Science

Note:
Only for Computer Science stream candidates.

1. Which data structure is commonly used to implement breadth first search in a graph?

(a) A queue ✓
(b) A stack
(c) A heap
(d) A hash table
(e) A splay tree

2. Let $G = (V, E)$ be an undirected simple graph. A subset $M \subseteq E$ is a *matching* in G if distinct edges in M do not share a vertex. A matching is *maximal* if no strict superset of M is a matching. How many maximal matchings does the following graph have?



(a) 1
(b) 2
(c) 3
(d) 4 ✓
(e) 5

3. Consider the problem of sorting n single digit integers (base 10). This problem can be solved in time

(a) $O(n \log n)$ but not $O(n \log \log n)$
(b) $O(n \log \log n)$ but not $O(n)$
(c) $O(n)$ but not $O(n/\log \log n)$ ✓
(d) $O(n/\log \log n)$
(e) None of the above.

4. Consider the following algorithm for computing the factorial of a positive integer n, specified in binary:

```
prod ← 1
for i from 1 to n
    prod ← prod × i
output prod
```

Assume that the number of bit operations required to multiply a k-bit positive integer with an ℓ-bit positive integer is at least $\Omega(k+l)$ and at most $O(kl)$. Then, the number of bit operations required by this algorithm is

(a) $O(n)$

(b) $O(n \log n)$ but $\omega(n)$

(c) $O(n^2)$ but $\omega(n \log n)$

(d) $O(n^3)$ but $\omega(n^2)$ ✓

(e) None of the above

5. There is an unsorted list of n integers. You are given 3 distinct integers and you have to check if all 3 integers are present in the list or not. The only operation that you are allowed to perform is a comparison. Let A be an algorithm for this task that performs the least number of comparisons. Let c be the number of comparisons done by A. Then,

(a) $c = 3n$

(b) $c = 2n + 5$

(c) $c \geq 3n - 1$

(d) $c \leq n$

(e) $c \leq 2n + 3$ ✓

6. We are given a graph G along with a matching M and a vertex cover C in it such that $|M| = |C|$. Consider the following statements:

(1) M is a maximum matching in G.

(2) C is a minimum vertex cover in G.

(3) G is a bipartite graph.

Which of the following is TRUE?

(a) Only statement (1) is correct

(b) Only statement (2) is correct

(c) Only statement (3) is correct

(d) Only statements (1) and (2) are correct ✓

(e) All the three statements (1), (2), and (3) are correct

7. Consider the following grammar: `P`, `Q`, `R` are non-terminals; `c`, `d` are terminals; `P` is the start symbol; and the production rules follow.

```
P ::= QR
Q ::= c
Q ::= RcR
R ::= ddQ
```

Which of the following is **False**:

(a) The length of every string produced by the grammar is even

(b) No string produced by the grammar has an odd number of consecutive `d`'s

(c) No string produced by the grammar has four consecutive `d`'s ✓

(d) No string produced by the grammar has three consecutive `c`'s

(e) Every string produced by the grammar has at least has many `d`'s as `c`'s

8. Let r_1 and r_2 be two regular expressions. They symbol $\equiv$ stands for equivalence of two regular expressions in the sense that if $r_1 \equiv r_2$, then both regular expressions describe the same language. Which of the following is/are FALSE?

(i) $(r_1 r_2)^* r_1 \equiv r_1(r_2 r_1)^*$

(ii) $(r_1^* r_2)^* r_1^* \equiv (r_1 + r_2)^*$

(iii) $(r_1^* r_2^*)^* \equiv (r_1 + r_2)^*$

(a) Only (i) is false

(b) Only (ii) is false

(c) Only (iii) is false

(d) Both (i) and (iii) are false

(e) None of the above ✓

9. Let $n \geq 2$ be any integer. Which of the following statements is FALSE?

(a) $n!$ divides the product of any n consecutive integers

(b) $\sum_{i=0}^{n} \binom{n}{i} = 2^n$

(c) $\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1}$, where $1 \leq i \leq n-1$

(d) If n is an odd prime, then n divides $2^{n-1} - 1$

(e) n divides $\binom{2n}{n}$ ✓

10. Consider the assertions

(A1) Given a directed graph G with positive weights on the edges, two special vertices s and t, and an integer k - it is NP-complete to determine if G has an s-t path of length at most k.

(A2) P = NP.

Then, which of the following is true?

(a) A1 implies A2 and A2 implies A1 ✓

(b) A1 implies A2 and A2 does not imply A1

(c) A1 does not imply A2 and A2 implies A1

(d) A1 does not imply A2 and A2 does not imply A1

(e) None of the above.

11. Consider the following function `count`, that takes as input `a`, an array of integers, and `N`, the size of the array.

```
int count(int a[], int N) {
    int i, j, count_FN;
    count_FN = 0;
    for (i=1; i<N; i++) {
        j=i-1;
          while (a[j] > a[i]) {
             count_FN++;
             j--;
           }
       }
    return count_FN;
}
```

Further, let `count_IS` be the number of comparisons made by the insertion sort algorithm on the array `a`.

Which of the following statements is TRUE for some constant c?

(a) **For all `N` $\geq c$, there exists an array of size `N` for which `count_IS` $\geq N^2/c$, while `count_FN` $\leq cN$** ✓

(b) For all `N` $\geq c$, there exists an array of size `N` for which `count_FN` $\geq N^2/c$, while `count_IS` $\leq cN$

(c) For all `N` $\geq c$, for all arrays of size `N`, `count_FN` $\leq$ `count_IS` $\leq c \times$ `count_FN`

(d) For all `N` $\geq c$, for all arrays of size `N`, `count_FN` $\geq N^2/c$

(e) None of the above

12. Given an undirected graph G, an ordering σ of its vertices is called a *perfect ordering* if for every vertex v, the neighbours of v which precede v in σ form a clique in G.

Recall that given an undirected graph G, a *clique* in G is a subset of vertices every two of which are connected by an edge, while a *perfect colouring of G with k colours* is an assignment of labels from the set $\{1, 2, \ldots, k\}$ to the vertices of G such that no two vertices which are adjacent in G receive the same label.

Consider the following problems.

Problem Special-Clique
INPUT: An undirected graph G, a positive integer k, and a perfect ordering σ of the vertices of G.
OUTPUT: Yes, if G has a clique of size at least k, No otherwise.

Problem Special-Colouring
INPUT: An undirected graph G, a positive integer k, and a perfect ordering σ of the vertices of G.
OUTPUT: Yes, if G has a proper colouring with at most k colours, No otherwise.

Assume that P $\neq NP$. Which of the following statements is true?

(a) Both Special-Clique and Special-Colouring are undecidable

(b) Only Special-Clique is in P

(c) Only SPECIAL-COLOURING is in P

(d) Both SPECIAL-CLIQUE and SPECIAL-COLOURING are in P ✓

(e) Neither of SPECIAL-CLIQUE and SPECIAL-COLOURING is in P, but both are decidable

13. Consider a directed graph $G = (V, E)$, where each edge $e \in E$ has a positive edge weight c_e. Determine the appropriate choices for the blanks below so that the value of the following linear program is the length of the shortest directed path in G from s to t. (Assume that the graph has at least one path from s to t.)

$$\begin{array}{rrcll} \underline{(\textbf{blank 1})\text{imize}} & X_t & & & \\ \text{s.t.} & X_s & = & 0 & \\ & X_w - X_v & \underline{\;(\textbf{blank 2})\;} & c_e & (\text{for each edge } e = (v, w) \in E). \end{array}$$

(a) blank 1: max, **blank 2:** $\leq$ ✓

(b) blank 1: max, blank 2: $\geq$

(c) blank 1: min, blank 2: $\leq$

(d) blank 1: min, blank 2: $\geq$

(e) blank 1: min, blank 2: $=$

14. Let G be a directed graph (with no self-loops or parallel edges) with $n \geq 2$ vertices and m edges. Consider the $n \times m$ incidence matrix M of G, whose rows are indexed by the vertices of G and the columns by the edges of G. The entry $m_{v,e}$ is defined as follows.

$$m_{v,e} = \begin{cases} -1 & \text{if } e = (v, w) \text{ for some vertex } w, \\ +1 & \text{if } e = (u, v) \text{ for some vertex } u, \\ 0 & \text{otherwise.} \end{cases}$$

Suppose every vertex of G is reachable from a special source vertex of G. Then, what is the rank of M?

(a) $m - 1$

(b) $m - n + 1$

(c) $\lceil m/2 \rceil$

(d) $n - 1$ ✓

(e) $\lceil n/2 \rceil$

15. Let $\mathbb{R}$ denote the set of real numbers. Let $d \geq 4$ and $\alpha \in \mathbb{R}$. Let

$$S = \left\{ (a_0, a_1, \ldots, a_d) \in \mathbb{R}^{d+1} : \sum_{i=0}^{d} a_i \alpha^i = 0 \text{ and } \sum_{i=0}^{d} i a_i \alpha^{i-1} = 0 \right\}.$$

Then,

(a) S is finite or infinite depending on the value of α

(b) S is a 2-dimensional vector subspace of $\mathbb{R}^{d+1}$

(c) S is a d-dimensional vector subspace of $\mathbb{R}^{d+1}$

(d) S is a $(d-1)$-dimensional vector subspace of $\mathbb{R}^{d+1}$ ✓

(e) For each $(a_0, a_1, \ldots, a_d) \in S$, the function

$$x \mapsto \sum_{i=0}^{d} a_i x^i$$

has a local optimum at α

—End of CS Section—

(SS section begins in the following page)

Part C: Systems Science

Note:
Only for Systems Science stream candidates.

1. Suppose that a random variable X can take 5 values $\{1, 2, 3, 4, 5\}$ with probabilities that depend upon $n \geq 0$ and are given by

$$P(X = k) = \frac{e^{kn}}{e^n + e^{2n} + e^{3n} + e^{4n} + e^{5n}}$$

for $k = 1, 2, 3, 4, 5$. What can one say about the expectation $E[X]$ as $n \to \infty$?

(a) It increases to infinity as $n \to \infty$

(b) It equals 3 for all values of $n \geq 0$

(c) It converges to 1 as $n \to \infty$

(d) It converges to 5 as $n \to \infty$ ✓

(e) It converges to 0 as $n \to \infty$

2. Consider a coin flip game between Amar, Akbar and Anthony. A fair coin (so that heads and tails each have probability 0.5) is independently flipped five times. Amar wins if at least three consecutive draws of heads are observed in the five coin tosses. Akbar wins if at least three consecutive draws of tails are observed in the five coin tosses. Anthony wins if the other two do not win. What is the probability of Anthony winning?

(a) 9/16

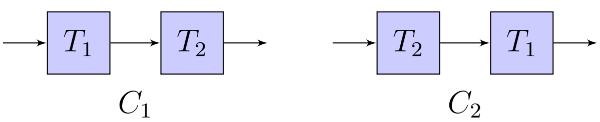
(b) 1/3

(c) 1/2 ✓

(d) 5/8

(e) 7/12

3. Consider two linear time invariant (LTI) systems T_1 and T_2 with impulse responses $h_1(n)$ and $h_2(n)$, respectively. Let there be two cascades C_1 and C_2, where in C_1, T_2 follows after T_1, and in C_2, T_1 follows after T_2.



Consider the following statements:

1. Both C_1 and C_2 are always LTI.
2. The impulse response of both C_1 and C_2 is the same.

3. Both C_1 and C_2 are LTI, only when h_1 and h_2 are causal.

Which of the following is TRUE?

(a) Only statement 1 is correct

(b) Only statement 3 is correct

(c) Both statements 1, 2 are correct ✓

(d) Both statements 2, 3 are correct

(e) None of the above

4. Evaluate the value of

$$\max\left(x^2 + (1-y)^2\right),$$

where the maximisation above is over x and y such that $0 \leq x \leq y \leq 1$.

(a) 0

(b) 2

(c) 1/2

(d) 1/4

(e) 1 ✓

5. Let Q be a unit square in the plane with corners at $(0,0)$, $(0,1)$, $(1,0)$ and $(1,1)$. Let B be a ball of radius 1 in the plane centered at the origin $(0,0)$. Let $Q+B$ denote the set of all vectors in the plane of the form $v+w$, where $v \in Q$ and $w \in B$. The area of $Q+B$ is:

(a) $5+\pi$ ✓

(b) $4+\pi$

(c) $3+\pi$

(d) $2+\pi$

(e) $1+\pi$

6. Consider a *degree*-5 *polynomial* function $f : (-\infty,\infty) \to (-\infty,\infty)$. If f exhibits at least four local maxima, which of the following is necessarily true? (Note : A local maximum is a point where the function value is the maximum in a sufficiently small neighbourhood.)

(a) $f(x) > 0, x \in (-\infty,\infty)$

(b) $f(50) < 0$

(c) The seventh derivative of $f(x)$ is negative for some $x \in [0,100]$

(d) f **has exactly 4 local maxima** ✓

(e) None of the above

7. Two players A and B of equal skill are playing a match. The first one to win 4 rounds wins the match. Both players are equally likely to win each round independent of the outcomes of the other rounds. After 3 rounds, A has won 2 rounds and B has won 1 round. Conditioned on this, what is the conditional probability that A wins the match?

(a) 5/8

(b) 2/3

(c) 11/16 ✓

(d) 5/7

(e) None of the above

8. Let a, b, c be real numbers such that the following system of equations has a solution

$$x + 2y + 3z = a \quad (1)$$
$$8x + 10y + 12z = b \quad (2)$$
$$7x + 8y + 9z = c - 1 \quad (3)$$

Let A be a matrix such that

$$A = \begin{bmatrix} a & 1 & c \\ b & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

What is the value of $\det(A)$?

(a) 1 ✓

(b) 2

(c) 3

(d) 4

(e) 5

9. Suppose you throw a dart and it lands uniformly at random on a target which is a disk of unit radius. What is the probability density function $f(x)$ of the distance of the dart from the center of the disk?

(a) $f(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(b) $f(x) = \begin{cases} 2x, & \textbf{if } 0 \le x \le 1 \\ 0, & \textbf{otherwise} \end{cases}$ ✓

(c) $f(x) = \begin{cases} 3x^2, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(d) $f(x) = \begin{cases} 4x^3, & \text{if } 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$

(e) None of the above.

10. Find the vector which is closest (in Euclidean distance) to $\begin{pmatrix}-1 & 1 & 1\end{pmatrix}$ which can be written in the form

$$a\begin{pmatrix}1 & 1 & 1\end{pmatrix} + b\begin{pmatrix}0 & 1 & -1\end{pmatrix}$$

where a and b are some real numbers. Recall that the (Euclidean) distance between two vectors $\begin{pmatrix}x_1 & x_2 & x_3\end{pmatrix}$ and $\begin{pmatrix}y_1 & y_2 & y_3\end{pmatrix}$ is given by $\sum_{i=1}^{3}(x_i - y_i)^2$.

(a) $\frac{1}{3}\begin{pmatrix}1 & 1 & 1\end{pmatrix}$ ✓

(b) $\frac{1}{2}\begin{pmatrix}0 & 1 & -1\end{pmatrix}$

(c) $\frac{1}{3}\begin{pmatrix}1 & 1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 1 & -1\end{pmatrix}$

(d) $-\frac{1}{3}\begin{pmatrix}1 & 1 & 1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0 & 1 & -1\end{pmatrix}$

(e) None of the above

11. A drunken man walks on a straight lane. At every integer time (in seconds) he moves a distance of 1 unit randomly, either forwards or backwards. What is the expectation of the square of the distance after 100 seconds from the initial position? Hint: The position at time 100 is a sum of independent and identically distributed random variables.

(a) 100 ✓

(b) $\frac{\sqrt{300}}{4}$

(c) 40

(d) 200

(e) 20π

12. An $n \times n$ matrix $\mathbf{P}$ is called a **Permutation Matrix** if each of its n columns and n rows contain exactly one 1 and $n-1$ 0's. Consider the following statements:

1. $\det(\mathbf{P})$ is either $+1$ or -1.
2. If λ is an eigen value of $\mathbf{P}$, then $|\lambda| = 1$.
3. $\mathbf{P}^T = \mathbf{P}^{-1}$.

Which of the following is TRUE?

(a) Only statement 1 is correct

(b) Only statements 1, 2 are correct

(c) Only statements 1, 3 are correct

(d) Only statements 2, 3 are correct

(e) All statements 1,2, and 3 are correct ✓

13. Calculate the minimum value attained by the function

$$\sin(\pi x) - \sqrt{2}\pi x^2$$

for values of x which lie in the interval $[0, 1]$.

(a) $\frac{1}{\sqrt{2}}\left(1-\frac{\pi}{8}\right)$

(b) 0

(c) $1-\frac{\pi}{2\sqrt{2}}$

(d) $-\frac{1}{\sqrt{2}}\left(1+\frac{9\pi}{2}\right)$

(e) $-\sqrt{2}\pi$ ✓

14. Let a bag contain ten balls numbered $1, 2, \ldots, 10$. Let three balls be drawn at random in sequence without replacement, and the number on the ball drawn on the i^{th} choice be $n_i \in \{1, 2, \ldots, 10\}$. What is the probability that $n_1 < n_2 < n_3$?

(a) $\frac{1}{3}$

(b) $\frac{1}{12}$

(c) $\frac{1}{4}$

(d) $\frac{1}{6}$ ✓

(e) None of the above

15. Consider the difference below for $m \geq 5$:

$$\sum_{n=1}^{m-1} \frac{1}{(1+n)^2} - \int_{x=1}^{m} \frac{1}{(1+x)^2} dx.$$

Which statement about the difference is TRUE?

(a) It is positive for infinitely many $m \geq 5$ and negative for infinitely many $m \geq 5$

(b) It is positive for all $m \geq 5$, and is increasing as m increases ✓

(c) It is negative for finitely many $m \geq 5$ and is positive for infinitely many m

(d) It is positive for all $m \geq 5$, and is decreasing as m increases

(e) It is negative for all $m \geq 5$

— END OF PAPER —